ISRAEL JOURNAL OF MATHEMATICS **TBD** (2024), 1–39
DOI: 10.1007/s11856-024-2633-x

CHAINS OF PATH GEOMETRIES ON SURFACES: THEORY AND EXAMPLES

BY

GIL BOR

Centro de Investigación en Matemáticas
A.P 402, Guanajuato, Gto. 36000, Mexico
e-mail: gil@cimat.mx

AND

TRAVIS WILLSE

Mathematics Department, Guilford College
5800 W Friendly Ave, Greensboro, NC 27410, USA
e-mail: twillse@guilford.edu

ABSTRACT

We derive the equations of chains for path geometries on surfaces by solving the equivalence problem of a related structure: sub-Riemannian geometry of signature $(1, 1)$ on a contact 3-manifold. This approach is significantly simpler than the standard method of solving the full equivalence problem for path geometry. We then use these equations to give a characterization of projective path geometries in terms of their chains (the chains projected to the surface coincide with the paths) and study the chains of four examples of homogeneous path geometries. In one of these examples (horocycles in the hyperbolic planes) the projected chains are bicircular quartics.

Received January 23, 2022 and in revised form July 19, 2022

1. Introduction

1.1. A quick reminder about path geometries on surfaces. A **path geometry** on a surface consists of a surface Σ (a 2-dimensional differentiable manifold) together with a non-degenerate 2-parameter family of unparametrized curves in Σ. (This definition will be reformulated below more abstractly and precisely; in particular, the non-degeneracy condition will be spelled out.) An **equivalence** of path geometries on two surfaces is a diffeomorphism of the surfaces which maps the paths of one surface onto those of the other. A **symmetry** of a path geometry on a surface is a self-equivalence.

The basic example is $\Sigma = \mathbb{R}P^2$ (the 2-dimensional real projective plane) equipped with the family of straight lines in it. A path geometry[1] which is locally equivalent to this example is called **flat**. A less obvious flat example is given by all parabolas whose focus is at the origin ('Kepler parabolas'; here $\Sigma := \mathbb{R}^2 \setminus 0$). It is doubly covered by straight lines via the (complex) quadratic map $z \mapsto z^2$.

Every path geometry is given locally by the graphs of solutions of a second-order ODE $y''=f(x,y,y')$. Conversely, a path geometry determines the ODE up to so-called point transformations, that is, changes of coordinates $(x,y)\mapsto(\tilde{x},\tilde{y})$. The flat example of straight lines in $\mathbb{R}P^2$ corresponds to $y''=0$. A path geometry is **projective** if its paths are the (unparametrized) geodesics of a torsion-free affine connection on Σ. Such path geometries correspond to ODEs $y''=f(x,y,y')$ where f is polynomial in y' of degree at most 3. Note that, somewhat surprisingly, this condition is independent of the coordinates x, y used on Σ. Thus a 'generic' path geometry is not projective, and in particular, non-flat. A non-projective example is the path geometry in $\mathbb{R}^2$ whose paths are all circles of a fixed radius.

A path geometry on a surface Σ defines a **dual** path geometry on the path space Σ^*, whose paths are parametrized by Σ: for each point $x \in \Sigma$ the corresponding path in Σ^* consists of all paths in Σ passing through x. Clearly, the dual of a flat path geometry is flat as well, an example of a self-dual path geometry. The path geometry of circles of fixed radius in $\mathbb{R}^2$ is an example of a self-dual non-projective path geometry. A projective path geometry is flat if and only if its dual is projective as well.

[1] We shall henceforth usually drop the qualifier "on a surface" since that is the only situation this article considers.

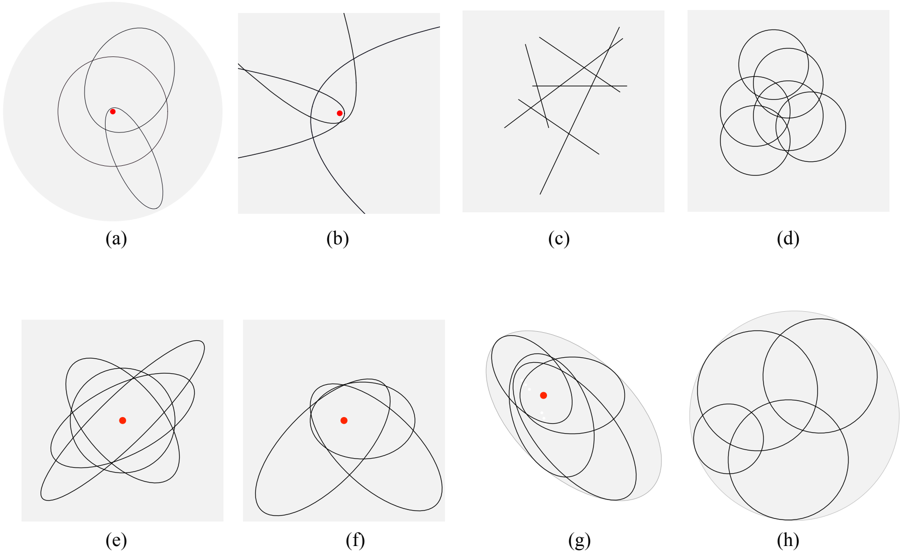

Figure 1. A gallery of 2D path geometries: (a) Kepler ellipses of fixed major axis. (b) Kepler parabolas. (c) Straight lines. (d) Circles of fixed radius. (e) Hooke ellipses of fixed area. (f) Kepler ellipses of fixed minor axis. (g) Kepler ellipses tangent to a fixed Kepler ellipse. (h) Circles tangent to a fixed circle (horocycles). Equivalence and duality relations among these geometries are: $a \approx e \approx f, b \approx c \approx g$ (flat), $a^* \approx h, b^* \approx b, d^* \approx d$.

A flat path geometry admits an 8-dimensional (local) group of symmetries (the projective group $\mathrm{PSL}_3(\mathbb{R})$), the maximum dimension possible for a path geometry. Conversely, a path geometry admitting an 8-dimensional local group of symmetries is necessarily flat (a theorem of Sophus Lie). The **sub-maximal** symmetry dimension, i.e., the maximum dimension of the local symmetry group of a non-flat path geometry, is 3. The path geometry of circles with a fixed radius is sub-maximal. Its symmetry group is the Euclidean group. Another sub-maximal example is given by central ellipses ('Hooke ellipses') of fixed area, where the symmetry group $\mathrm{SL}_2(\mathbb{R})$ acts by its standard linear action on $\mathbb{R}^2$ (here $\Sigma = \mathbb{R}^2 \setminus 0$). In contrast to the previous example of circles with fixed radius, this example is projective and non–self-dual: its dual is the hyperbolic plane and the paths are the horocycles (in the Poincaré disk or upper half-plane model horocycles are the circles tangent to the boundary; see Section 4.4 below).

The subject was studied extensively in the second half of the 19th century by Roger Liouville (a relative of the more famous Joseph Liouville), Sophus Lie and his student Arthur Tresse, who produced a local classification, over the complex numbers, of sub-maximal path geometries (i.e., those admitting a 3-dimensional group of symmetries) [25]. This classification has been since refined over the real numbers [14]. The only non-flat projective items on the list are the above mentioned case of Hooke ellipses of fixed area and Hooke hyperbolas of fixed discriminant (see Table 2 in the Appendix of [3] for several equivalent models of these path geometries).

1.2. An abstract reformulation of path geometry. We describe here briefly a more abstract and rigorous reformulation of path geometries on surfaces, useful also for introducing chains. For further details we recommend V. I. Arnol'd's book [1, Chapter 1, Section 6].

Given a surface Σ, let $\mathbb{P}T\Sigma$ be the (3-dimensional) total space of its projectivized tangent bundle. That is, a point in $\mathbb{P}T\Sigma$ corresponds to a point in Σ together with a tangent line at the point (a 1-dimensional linear subspace of the tangent space at the point). There is a standard contact distribution D on $\mathbb{P}T\Sigma$, given by the 'skating' condition: "the point moves along the line", or "the line rotates about the point." The fibers of the base point projection $\mathbb{P}T\Sigma \to \Sigma$ are integral curves of D. Their tangents form the vertical line field $L_1 \subset D$. A path $\gamma \subset \Sigma$ is lifted to $\mathbb{P}T\Sigma$ by mapping a point on γ to the tangent line to γ at this point. The lifted curve is clearly an integral curve of D, as it satisfies the skating condition. The non-degeneracy assumption on a path geometry on Σ is that the lifted paths form a smooth 1-dimensional foliation of $\mathbb{P}T\Sigma$, transverse to L_1 (in D); equivalently, the tangent lines to the lifted curves form a smooth line field $L_2 \subset D$, complementary to L_1, so that

$$D = L_1 \oplus L_2.$$

We thus arrive at an abstract reformulation of a path geometry:

Definition 1.1: A (2-dimensional) path geometry is a smooth 3-manifold M together with an (ordered) pair of smooth line fields $L_1, L_2 \subset TM$, spanning a contact distribution $D = L_1 \oplus L_2$. The path geometry dual to (M, L_1, L_2) is (M, L_2, L_1).

Remark 1.2: Another common name for (M, L_1, L_2) is a **para-CR structure**, due to the formal similarity with a (Levi-non-degenerate) CR structure (M, D, J). The latter is a contact distribution D on a 3-manifold M together with a complex structure $J \in \mathrm{End}(D)$, i.e., $J^2 = -\mathrm{id}_D$; equivalently, it is a splitting $D \otimes \mathbb{C} = D^{1,0} \oplus D^{0,1}$, the direct sum of a conjugate pair of complex line bundles (the $\pm i$-eigenbundles of $J \otimes \mathbb{C}$).

In the real-analytic setting, CR and para-CR structures have a common complexification: a complex 3-manifold together with a pair of (complex) line fields spanning a (complex) contact distribution.

Remark 1.3: Some authors define a path geometry as a 2-parameter family of curves on a surface Σ, a unique curve through any given point of Σ in any given direction (see, e.g., the first paragraph of [15], or the "fancy formulation" of [19, Section 8.6]). Definition 1.1 is more precise and general: first, the surface Σ is recovered from (M, L_1, L_2) as the space of integral curves of L_1, which may exist as a smooth surface only locally. Second, even if Σ exists, the set of directions at a given $x \in \Sigma$ for which a curve exists may be only an open subset in $\mathbb{P}T_x\Sigma$. For example, for the path geometry of Hooke (or central) ellipses in $\mathbb{R}^2 \setminus 0$ a curve exists only in non-radial directions. Third, there may be more than one curve in a given direction. For example, for circles of fixed radius in $\mathbb{R}^2$, there are two circles passing through each point in a given direction. This can be remedied by considering instead **oriented** circles of fixed radius and the **spherized** tangent bundle $\mathbb{S}T\mathbb{R}^2$ (that is, $T\mathbb{R}^2$, with the zero section removed, mod $\mathbb{R}^+$) instead of $\mathbb{P}T\mathbb{R}^2$. An analogous remedy applies to the aforementioned path geometry of horocycles in the hyperbolic plane.

We shall not dwell here further on these details and refer the interested reader to [6, Sections 4.2.3 and 4.4.3], where our notion of a path geometry on a surface is called both a generalized path geometry and a Lagrangean contact structure on a 3-manifold; the two notions differ in higher dimension.

1.3. Chains of path geometries via the Fefferman metric.

In the CR case there is a well-known, naturally associated 4-parameter family of curves on M, called chains, one chain for each given point in M in a given direction transverse to the contact distribution. They are considered the CR analog of geodesics in Riemannian geometry (see the recent article [12] for a variational formulation). Chains were introduced by É. Cartan while solving the equivalence problem of CR geometry [10, 20] and were studied extensively by many authors, such as Chern–Moser [13] and C. Fefferman [17], who showed that they arise from a natural construction, considerably simpler than Cartan's, nowadays called the Fefferman metric: a conformal Lorentzian metric, i.e., of signature $(3, 1)$, defined on the total space of a certain circle bundle over M. The chains of the CR structure are then the projections onto M of the non-vertical null geodesics of the Fefferman metric.

Similarly, to each path geometry (M, L_1, L_2) one can associate a natural 4-parameter family of curves on M, a unique curve through any given point in M in any given direction transverse to the contact distribution $D := L_1 \oplus L_2$. The study of this natural class of curves is quite recent. The earliest reference we know of is a 2005 article of A. Čap and V. Žádník [7] (path geometries on surfaces appear there in Section 2 as 3-dimensional Lagrangean contact structures). See also [6, Sections 5.3.7–8 and 5.3.13–14]. Both references define chains using the associated Cartan geometry. However, as in the CR case, there is a significant shortcut via the Fefferman metric. This is a conformal metric of signature $(2, 2)$ on the total space of an $\mathbb{R}^*$-bundle over M, and the chains are the projections onto M of non-vertical null geodesics of the Fefferman metric. In this article we explain this construction and use it to give several concrete examples.

Remark 1.4: As mentioned in Remark 1.3, path geometries on surfaces generalize in higher dimension to both (generalized) path geometries and Lagrangean contact geometries. The Fefferman-type construction of a conformal structure described here generalizes in higher dimensions to Lagrangean contact structures but not to path geometries.

1.4. Contents of the article. In the next section we re-derive, as a warm-up and a reminder, the Fefferman metric for a CR structure (M, D, J). The construction appeared first in Fefferman's article [17] for a CR manifold embedded as a real hypersurface in a complex manifold, followed by intrinsic constructions, first direct ones in [16, 22], then more advanced constructions that use the full solution of the equivalence problem for CR structures (Cartan bundle and connection), such as [5, 7, 23]. We view instead a CR structure as a conformal class of sub-Riemannian geometries of contact type, solve the equivalence problem of sub-Riemannian geometries of contact type following [18]—which is much simpler than that for CR geometry, use a sub-Riemannian metric on D to define a Lorentzian metric on $\mathbb{S}D$ (the spherization of D), then show that conformally equivalent sub-Riemannian metrics on D induce conformally equivalent Lorentzian metrics on $\mathbb{S}D$. In retrospect, our construction can be regarded as a shorter version of [16, 22], using [18]. It is still too complicated conceptually for our taste, and the formula (9) below for the metric appears a bit like magic, but this method is the best we have so far and is quite easy to work with.

Once the construction of the Fefferman metric for CR geometry is understood, we construct in Section 3 in a similar fashion the Fefferman metric for a path geometry. As far as we know, our derivation is new, and before this article the only available construction of the Fefferman metric for path geometry has been via the solution of the full equivalence problem for such a structure (see, e.g., [7, 23]), which is considerably more involved than our derivation.

In Section 3.2 we prove the following theorem, apparently new:

THEOREM 1: *A path geometry on a surface Σ is projective if and only if the chains on $\mathbb{P}T\Sigma$ project to the paths in Σ.*

In the last section we study in some detail the chains of four homogeneous path geometries mentioned above: straight lines, circles of fixed radius, central ellipses of fixed area and horocycles in the hyperbolic plane.

ACKNOWLEDGMENTS. We thank the anonymous reviewer for some useful suggestions. GB acknowledges support from CONACYT Grant A1-S-45886. TW is grateful for support and hospitality from CIMAT during an extended visit in the 2019–20 academic year and for support from Guilford College.

2. The Fefferman metric for CR 3-manifolds (revisited)

Let (M, D, J) be a CR 3-manifold, i.e., $D \subset TM$ is a contact 2-distribution (that is, $[D, D] = TM$) and $J \in \mathrm{End}(D)$ satisfies $J^2 = -\,\mathrm{id}_D$. Canonically associated to the CR structure is a circle bundle $\mathbb{S}D \to M$, the 'spherization' of D, with a conformal class of metrics of signature $(3,1)$ on $\mathbb{S}D$, the Fefferman metric. It depends on the second-order jet of the CR structure, so is not so easy to see. The fibers of $\mathbb{S}D \to M$ are null geodesics, and the projections of the non-vertical null geodesics to M are the **chains** of the CR structure, forming a 4-parameter family of curves on M.

THE CONSTRUCTION. Fix a positive contact form η^3 on M, i.e., a 1-form satisfying

$$D = \mathrm{Ker}(\eta^3), \tag{1}$$

$$\mathrm{d}\eta^3(X, JX) > 0 \quad \text{for every } X \in D,\ X \neq 0. \tag{2}$$

Remark 2.1: A general contact manifold does not admit necessarily a global contact form (a 1-form whose kernel is D) but the contact structure of a CR manifold does, using the orientation of D induced by J. If M is connected then any global contact form is either positive or negative.

Recall that the coframe bundle $\pi : F^* \to M$ is the principal $\mathrm{GL}_3(\mathbb{R})$-bundle whose fiber at a point $x \in M$ consists of all linear isomorphisms $u : T_xM \to \mathbb{R}^3$. The **tautological 1-form** on F^* is the $\mathbb{R}^3$-valued 1-form ω whose value at $u \in F^*$ is $u \circ (\mathrm{d}\pi)_u$.

Now a positive contact form η^3 on M defines a positive-definite inner product on D, $\langle X, Y\rangle := \mathrm{d}\eta^3(X, JY)$. An **adapted coframe** is an extension of η^3 to a coframe $\eta = (\eta^1, \eta^2, \eta^3)^t$ (we view elements of $\mathbb{R}^3$ as column vectors), satisfying

$$\mathrm{d}\eta^3 = \eta^1 \wedge \eta^2, \tag{3}$$

$$\langle \cdot, \cdot \rangle = [(\eta^1)^2 + (\eta^2)^2]|_D. \tag{4}$$

It is easy to show that for a fixed η^3 these 2 equations define a circle's worth of coframes at each $x \in M$. Thus, let $S^1 \subset \mathrm{GL}_3(\mathbb{R})$ be the set of matrices of the form

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $B \subset F^*$ the set of coframes adapted to η^3. Then $B \to M$ is a principal S^1-subbundle, an S^1-reduction of F^*, whose local sections consist of adapted coframes.

We continue to denote by $\omega = (\omega^1, \omega^2, \omega^3)^t$ the restriction of the tautological 1-form on F^* to B. Then there are a unique 1-form α and functions a_1, a_2 on B such that

$$\mathrm{d}\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = -\begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & -a_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 \end{pmatrix}. \tag{5}$$

(See [18, equation (1)].) Furthermore, there are unique functions b_1, b_2, K on B such that

$$\mathrm{d}\alpha = b_1\omega^2 \wedge \omega^3 + b_2\omega^3 \wedge \omega^1 + K\omega^1 \wedge \omega^2. \tag{6}$$

(See [18, equation (4)]; in fact, K descends to M. Also, α is essentially the Webster connection form [26], a_1, a_2 its torsion, and K the Webster scalar curvature.)

Define a Lorentzian metric on B by

$$\mathrm{g} := \omega^1 \cdot \omega^1 + \omega^2 \cdot \omega^2 + \omega^3 \cdot \sigma, \tag{7}$$

where σ is a 1-form, to be determined later, and $\cdot$ is the symmetric product of 1-forms.

Let $\mathbb{S}D$ be the 'spherization' (or 'ray projectivization') of D, i.e., the quotient of D with the zero section removed, by the dilation action of $\mathbb{R}^+$. There is an obvious S^1-action on D, commuting with the $\mathbb{R}^+$-action, thus making $\mathbb{S}D$ a principal S^1-bundle. Note that $\mathbb{S}D$, unlike B, is canonically associated to (M, D, J): to define B we needed to choose the positive contact form η^3. Define an isomorphism of principal S^1-bundles

$$h : B \to \mathbb{S}D, \quad u \mapsto [u^{-1}\mathbf{e}_1], \tag{8}$$

where $\mathbf{e}_1 = (1, 0, 0)^t$. That is, $h(u) = [X] \in \mathbb{S}D$, where $X \in D$ is the unique vector in T_xM, $x = \pi(u)$, satisfying $u^1(X) = 1, u^2(X) = u^3(X) = 0$. We then use h to map the Lorentzian metric on B of equation (7) to a Lorentzian metric on $\mathbb{S}D$. In general, for arbitrary σ in formula (7), the resulting metric on $\mathbb{S}D$ depends on the choice of η^3 in a complicated way, but for a careful choice of σ the conformal class of the Lorentzian metric on $\mathbb{S}D$ is independent of the choice of η^3.

Remark 2.2: There are other models for the underlying space of the Fefferman metric instead of $\mathbb{S}D$ (a matter of taste). For example, one can take the spherization of the dual bundle $D^\vee$, in which case the formula for the identification $B \to \mathbb{S}D^\vee$ is a little simpler: $u \mapsto [u^1|_D]$. Another model is the spherization of the canonical bundle $\Lambda^{2,0}D \subset \Lambda^2 T^*D \otimes \mathbb{C}$, as in [22]; the identification with B in this case is $u \mapsto [u^3 \wedge (u^1 + iu^2)|_D]$. Also, the metric on $\mathbb{S}D$ is invariant under the antipodal map in each fiber (a circle), and so it descends to the (full) projectivization $\mathbb{P}D$.

The following theorem is Theorem (3.8) of [22], or the theorem on page 41 of [16]. Our proof is essentially that of [22], reformulated so as to facilitate its extension in the next section to path geometries.

THEOREM 2: *Let* (M, D, J) *be a CR 3-manifold,* $\mathbb{S}D \to M$ *the spherization of* D *and* η^3 *any positive contact* 1*-form, as in equations* (1) *and* (2)*. Define a* 1*-form* σ *on the total space of the associated circle bundle* $B \to M$*,*

$$\sigma = \frac{4}{3}\alpha - \frac{1}{3}K\omega^3, \tag{9}$$

where α, K *are defined via equations* (5) *and* (6)*. Then the conformal class of the Lorentzian metric induced on* $\mathbb{S}D$ *by equation* (7)*, via the isomorphism* (8)*, is independent of the choice of* η^3*. In fact, multiplying* η^3 *by a positive function rescales the induced metric on* $\mathbb{S}D$ *by the same factor.*

Proof. If η^3 is a positive contact form on M, then any other positive contact form is of the form $\tilde{\eta}^3 = \lambda^2\eta^3$, for some positive function $\lambda : M \to \mathbb{R}^+$. Changing η^3 to $\tilde{\eta}^3$ changes B to $\tilde{B}$, another S^1-reduction of the coframe bundle of M, with corresponding metric $\tilde{\mathrm{g}}$ and isomorphism $\tilde{h} : \tilde{B} \to \mathbb{S}D$. We thus need to show that the composition $f := \tilde{h}^{-1} \circ h : B \to \tilde{B}$ satisfies $f^*(\tilde{g}) = \lambda^2 g$.

Let us pull-back λ to B by the projection $B \to M$, denoting the result by λ as well. Then

$$\mathrm{d}\lambda = \lambda_i\omega^i, \quad \mathrm{d}\lambda_i = \lambda_{i0}\alpha + \lambda_{ij}\omega^j, \tag{10}$$

for some functions $\lambda_i, \lambda_{ij}, \lambda_{i0}$ on B, $1 \le i, j \le 3$. (Note that by definition λ descends to M; in general the λ_i do not, but λ_3 does.)

LEMMA 2.3:

$$\lambda_{10} = -\lambda_2, \quad \lambda_{20} = \lambda_1, \quad \lambda_{12} - \lambda_{21} = \lambda_3.$$

Proof. These identities follow immediately from expanding $\mathrm{d}(\mathrm{d}\lambda) = 0$. ∎

Now a section $\eta = (\eta^1, \eta^2, \eta^3) : M \to B$ of $B \to M$ is a coframe adapted to η^3, so $f \circ \eta : M \to \tilde{B}$ is a section of $\tilde{B} \to M$, a coframe adapted to $\tilde{\eta}^3 = \lambda^2\eta^3$.

LEMMA 2.4: $f \circ \eta = \Lambda\eta$*, where*

$$\Lambda = \begin{pmatrix} \lambda & 0 & -2\lambda_2 \\ 0 & \lambda & 2\lambda_1 \\ 0 & 0 & \lambda^2 \end{pmatrix}.$$

Proof. It is enough to check that $\tilde{\eta} := \Lambda\eta$ satisfies equations (1)–(4) above, with $\tilde{\eta}^3 = \lambda^2\eta^3$ instead of η^3, as well as $\tilde{\eta}^1(\tilde{X}) = 1, \tilde{\eta}^2(\tilde{X}) = \tilde{\eta}^3(\tilde{X}) = 0$ for $\tilde{X} = X/\lambda$. ∎

LEMMA 2.5: $f^*\omega_{\tilde{B}} = \Lambda\omega_B$.

Proof. Let $\eta \in B$, $\tilde{\eta} = f(\eta)$. By Lemma 2.4, $\tilde{\eta} = \Lambda\eta$, hence

$$\begin{aligned}(f^*\omega_{\tilde{B}})_\eta &= (\omega_{\tilde{B}})_{\tilde{\eta}} \circ (\mathrm{d}f)_\eta = \tilde{\eta} \circ (\mathrm{d}\tilde{\pi})_{\tilde{\eta}} \circ (\mathrm{d}f)_\eta \\ &= \tilde{\eta} \circ \mathrm{d}(\tilde{\pi} \circ f)_\eta = \tilde{\eta} \circ (\mathrm{d}\pi)_\eta = \Lambda\eta \circ \mathrm{d}\pi_\eta = \Lambda(\omega_B)_\eta.\end{aligned}$$ ∎

Notation: For sake of readability, we adopt henceforth the following abbreviated notation:

$$\omega := \omega_B, \quad \mathrm{g} := \mathrm{g}_B, \ldots, \tilde{\omega} := f^*\omega_{\tilde{B}}, \quad \tilde{\mathrm{g}} := f^*\mathrm{g}_{\tilde{B}}, \ldots, \text{ etc.}$$

Thus, for example, Lemma 2.5 reads $\tilde{\omega} = \Lambda\omega$.

We proceed with the proof of Theorem 2. It is clearly enough to show an infinitesimal version of the claimed conformal invariance. Suppose $\lambda = \lambda(t)$ is differentiable and that it satisfies $\lambda(0) = 1$. Denote by a dot the derivative with respect to t at $t = 0$ of objects on $\tilde{B}$ pulled back to B by f, e.g., $\dot{\lambda} = \lambda'(0)$, $\dot{\lambda}_i = \lambda_i'(0)$, $\dot{\lambda}_{ij} = \lambda_{ij}'(0)$, $\dot{\mathrm{g}} = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\tilde{\mathrm{g}}$, etc. Then $\tilde{\mathrm{g}} = \lambda^2\mathrm{g}$ if and only if $\dot{\mathrm{g}} = 2\dot{\lambda}\mathrm{g}$ (for all η^3 and $\lambda(t)$ satisfying $\lambda(0) = 1$). Now calculate using the previous lemmas:

$$\dot{\omega}^1 = \dot{\lambda}\omega^1 - 2\dot{\lambda}_1\omega^3, \quad \dot{\omega}^2 = \dot{\lambda}\omega^2 + 2\dot{\lambda}_2\omega^3, \quad \dot{\omega}^3 = 2\dot{\lambda}\omega^3,$$
$$\dot{\mathrm{g}} = 2\dot{\lambda}\mathrm{g} + (\dot{\sigma} - 4\dot{\lambda}_2\omega^1 + 4\dot{\lambda}_1\omega^2) \cdot \omega^3.$$

Thus $\dot{\mathrm{g}} = 2\dot{\lambda}\mathrm{g}$ if and only if

$$\dot{\sigma} = 4(\dot{\lambda}_2\omega^1 - \dot{\lambda}_1\omega^2). \tag{11}$$

To calculate $\dot{\sigma}$, using formula (9), we need formulas for $\dot{\alpha}$ and $\dot{K}$. To find $\dot{\alpha}$ we find first a formula for $\tilde{\alpha}$. Write the structure equations (5) for $\tilde{\omega}$, substitute $\tilde{\omega} = \Lambda\omega$, and equate coefficients. The result is

$$\tilde{\alpha} = \alpha + 3\frac{\lambda_2}{\lambda}\omega^1 - 3\frac{\lambda_1}{\lambda}\omega^2 - \Big[3\frac{(\lambda_1)^2 + (\lambda_2)^2}{\lambda^2} + \frac{\lambda_{11} + \lambda_{22}}{\lambda}\Big]\omega^3.$$

Taking derivative with respect to t at $t = 0$ of the last formula, we get

$$\dot{\alpha} = 3\dot{\lambda}_2\omega^1 - 3\dot{\lambda}_1\omega^2 - (\dot{\lambda}_{11} + \dot{\lambda}_{22})\omega^3.$$

To find $\dot{K}$ there is a shortcut, avoiding an explicit formula for $\tilde{K}$, by noting first that K is defined by $\mathrm{d}\alpha \equiv K\omega^1 \wedge \omega^2 \pmod{\alpha, \omega^3}$. Taking d of the above formula for $\dot{\alpha}$, we get, using equations (10), $\mathrm{d}\dot{\alpha} \equiv -4(\dot{\lambda}_{11} + \dot{\lambda}_{22})\omega^1 \wedge \omega^2$ $\pmod{\alpha, \omega^3}$. Taking derivative with respect to t of $\mathrm{d}\tilde{\alpha} \equiv \tilde{K}\tilde{\omega}^1 \wedge \tilde{\omega}^2 \pmod{\tilde{\alpha}, \tilde{\omega}^3}$, we get $\mathrm{d}\dot{\alpha} \equiv (\dot{K} + 2\dot{\lambda}K)\omega^1 \wedge \omega^2 \pmod{\alpha, \omega^3}$, hence

$$\dot{K} = -2\dot{\lambda}K - 4(\dot{\lambda}_{11} + \dot{\lambda}_{22}).$$

Now using the above expressions for $\dot{\alpha}$, $\dot{K}$ and $\dot{\omega}^3$, we obtain from equation (9)

$$\dot{\sigma} = \frac{4}{3}\dot{\alpha} - \frac{1}{3}(\dot{K}\omega^3 + K\dot{\omega}^3) = 4(\dot{\lambda}_2\omega^1 - \dot{\lambda}_1\omega^2),$$

as needed. ∎

2.1. Example: left-invariant CR structures on SU_2. The left-invariant $\mathfrak{su}_2$-valued Maurer–Cartan form on SU_2 is

$$\Theta = g^{-1}\mathrm{d}g = \begin{pmatrix} i\theta^1 & \theta^2 + i\theta^3 \\ -\theta^2 + i\theta^3 & -i\theta^1 \end{pmatrix}. \tag{12}$$

The Maurer–Cartan equation $\mathrm{d}\Theta = -\Theta \wedge \Theta$ gives

$$\mathrm{d}\theta^1 = -2\theta^2 \wedge \theta^3, \quad \mathrm{d}\theta^2 = -2\theta^3 \wedge \theta^1, \quad \mathrm{d}\theta^3 = -2\theta^1 \wedge \theta^2. \tag{13}$$

For each $t \in [1, \infty)$ let

$$\eta^1 = \sqrt{t}\,\theta^1, \quad \eta^2 = \theta^2/\sqrt{t}, \quad \eta^3 = -\theta^3/2.$$

One can show that every left-invariant CR structure $D^{0,1} \subset T\mathrm{SU}_2 \otimes \mathbb{C}$ is equivalent (via right translation), for a unique $t \in [1, \infty)$, to $\{\eta^1 + i\eta^2, \eta^3\}^\perp$. For $t = 1$ we obtain the standard 'spherical' CR structure on $\mathrm{SU}_2 \simeq S^3$. For $t > 1$ these are non-spherical CR structures. Distinct t determine inequivalent structures (see [3], Prop. 5.1). We use (13) to find

$$\mathrm{d}\eta^1 = 4t\,\eta^2 \wedge \eta^3, \quad \mathrm{d}\eta^2 = (4/t)\eta^3 \wedge \eta^1, \quad \mathrm{d}\eta^3 = \eta^1 \wedge \eta^2.$$

Using this coframe we identify $B \simeq \mathrm{SU}_2 \times S^1$ and $\omega = \bar{u} \cdot \eta$, where $u = e^{i\varphi}$. Explicitly,

$$\omega^1 = \sqrt{t}(\cos\theta)\theta^1 + \frac{1}{\sqrt{t}}(\sin\theta)\theta^2, \quad \omega^2 = -\sqrt{t}(\sin\theta)\theta^1 + \frac{1}{\sqrt{t}}(\cos\theta)\theta^2, \quad \omega^3 = -\frac{1}{2}\theta^3.$$

Inserting these into equations (5)–(6), we obtain

$$\alpha = \theta^4 - \Big(t + \frac{1}{t}\Big)\theta^3, \quad K = 2\Big(t + \frac{1}{t}\Big),$$

where $\theta^4 := \mathrm{d}\varphi$. Inserting all this into equations (7)–(9), we get

$$\mathrm{g} = t(\theta^1)^2 + \frac{1}{t}(\theta^2)^2 + \frac{1}{2}\Big(t + \frac{1}{t}\Big)(\theta^3)^2 - \frac{2}{3}\theta^3 \cdot \theta^4.$$

This is essentially formula (15) of [11]; the coefficient of our $\theta^3 \cdot \theta^4$ term can be made to agree with the cited formula by rescaling the φ coordinate by a constant. See also [11] for a study of the chains of this example via null geodesics of the Fefferman metric.

3. The Fefferman metric for path geometries

Let (M, L_1, L_2) be a path geometry, i.e., L_1, L_2 is a pair of line fields on a 3-manifold M, spanning a contact distribution $D := L_1 \oplus L_2$. Let us fix a contact form η^3, that is,

$$D = \mathrm{Ker}(\eta^3)$$

(possibly defined only locally, see Remark 3.1 below). An **adapted coframe** (with respect to η^3) is an extension of η^3 to a (local) coframe (η^1, η^2, η^3) satisfying

$$\mathrm{d}\eta^3 = \eta^1 \wedge \eta^2, \tag{14}$$

$$\eta^1|_{L_2} = \eta^2|_{L_1} = 0. \tag{15}$$

These equations define an $\mathbb{R}^*$-structure, i.e., an $\mathbb{R}^*$-principal subbundle $B \subset F^*$, whose local sections are the coframes adapted to η^3, where $s \in \mathbb{R}^*$ acts by

$$(\eta^1, \eta^2, \eta^3) \mapsto (\eta^1/s, s\eta^2, \eta^3).$$

Let $D^* = D \setminus (L_1 \cup L_2) \subset D$, with spherization $\mathbb{S}D^* \subset \mathbb{S}D$. The Fefferman metric associated to the path geometry is a conformal pseudo-Riemannian metric of signature $(2,2)$ on $\mathbb{S}D^*$. We shall define it in a manner similar to the CR case. The splitting $D = L_1 \oplus L_2$ defines an involution $J \in \mathrm{End}(D)$, $J^2 = \mathrm{id}$, by

$$J(X_1 + X_2) = X_1 - X_2, \quad X_i \in L_i,\ i = 1, 2. \tag{16}$$

The contact form η^3 defines on D an area form, $\mathrm{d}\eta^3|_D$, and an indefinite metric of signature $(1,1)$,

$$\langle X, Y\rangle := \mathrm{d}\eta^3(X, JY).$$

Now $D^* = D^+ \cup D^-$, where $D^\pm$ are the positive (resp. negative) vectors with respect to $\langle\cdot,\cdot\rangle$, and corresponding decomposition $\mathbb{S}D^* = \mathbb{S}D^+ \cup \mathbb{S}D^-$. Both $\mathbb{S}D^\pm$

are $\mathbb{R}^*$-principal bundles over M, where $s \in \mathbb{R}^*$ acts by

$$[X_1 + X_2] \mapsto [sX_1 + X_2/s], \quad X_i \in L_i.$$

Note that $D^\pm$ are interchanged by J or by taking $-\eta^3$ instead of η^3. There is an identification of $\mathbb{R}^*$-principal bundles,

$$h : B \to \mathbb{S}D^+, \quad u \mapsto [X], \quad \text{where } u^1(X) = u^2(X) = 1,\ u^3(X) = 0. \tag{17}$$

We shall define a pseudo-Riemannian metric of signature $(2,2)$ on B, map it by h to $\mathbb{S}D^+$, then by J to $\mathbb{S}D^-$. As in the CR case, we show that the associated conformal class of metrics on $\mathbb{S}D^*$ is independent of the chosen contact form η^3.

Remark 3.1: A general contact 3-manifold is naturally oriented. The Lie bracket of sections of D defines an isomorphism $\Lambda^2(D) \to TM/D$, but these isomorphic line bundles need not be trivial, i.e., there might not exist on M a global contact form (a non-vanishing section of $D^\perp \simeq (TM/D)^*$). In the CR case, J defines an orientation of $D \subset TM$, hence of TM/D as well (since TM is oriented), and a dual orientation of $D^\perp = (TM/D)^*$, so there is always a global contact form. This is not the case for a path geometry (e.g., $M = \mathbb{P}T\mathbb{R}^2$, equipped with the standard flat path geometry). But this topological difficulty is minor, we can still define B locally, then show that the conformal structures defined on $\mathbb{S}D^*$ restricted to open subsets of M agree on intersections. We shall not dwell on the details.

We shall now proceed with the plan outlined above, in the paragraph before Remark 3.1.

The structure equations for any $\mathbb{R}^*$-connection form α on $B \to M$ are

$$\mathrm{d}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} = -\begin{pmatrix}\alpha & 0 & 0\\ 0 & -\alpha & 0\\ 0 & 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} + \begin{pmatrix}T^1_{23} & T^1_{31} & T^1_{12}\\ T^2_{23} & T^2_{31} & T^2_{12}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\omega^2\wedge\omega^3\\ \omega^3\wedge\omega^1\\ \omega^1\wedge\omega^2\end{pmatrix},$$

where T^i_{jk} are some real functions on B (the coefficients of the torsion tensor of the connection). Starting from any such connection it is easy to show that it can be modified, in a unique way, by adding to α multiples of the ω^i, so as to render

$$T^1_{31} = T^1_{12} = T^2_{12} = 0$$

(in fact doing so also solves the equivalence problem for path geometry equipped with a fixed contact form). Taking the exterior derivative of $\mathrm{d}\omega^3 = \omega^1 \wedge \omega^2$ shows that $T^2_{23} = 0$ as well. The structure equations now become

$$\begin{aligned} \mathrm{d}\omega^1 &= -\alpha \wedge \omega^1 + a_1 \omega^2 \wedge \omega^3, \\ \mathrm{d}\omega^2 &= \alpha \wedge \omega^2 + a_2 \omega^3 \wedge \omega^1, \\ \mathrm{d}\omega^3 &= \omega^1 \wedge \omega^2, \end{aligned} \tag{18}$$

for some functions a_1, a_2 on B. Taking exterior derivative of these equations we get

$$\mathrm{d}\alpha = b_1 \omega^2 \wedge \omega^3 + b_2 \omega^3 \wedge \omega^1 + K \omega^1 \wedge \omega^2 \tag{19}$$

for some functions b_1, b_2, K on B (i.e., $\mathrm{d}\alpha$ is semi-basic, containing no $\alpha \wedge \omega^i$ terms).

Theorem 3: *Let (M, L_1, L_2) be a path geometry and $\mathbb{S}D^* \subset \mathbb{S}D$ the set of rays in $D = L_1 \oplus L_2$ not contained in $L_1 \cup L_2$. Then there is a canonically associated conformal class of metrics of signature $(2,2)$ on $\mathbb{S}D^*$, called the Fefferman metric, defined as follows. Associated with each contact 1-form η^3 on M is an $\mathbb{R}^*$-reduction $B \to M$ of the coframe bundle of M, given by equations (14)–(15), a unique $\mathbb{R}^*$-connection form α on B satisfying equations (18) and the 1-form*

$$\sigma := -\frac{2}{3}\alpha + \frac{1}{6}K\omega^3, \tag{20}$$

where K is defined via equations (19), and where $\omega^1, \omega^2, \omega^3$ are the tautological 1-forms on the coframe bundle of M restricted to B. Then

$$\mathrm{g} := \omega^1 \cdot \omega^2 + \omega^3 \cdot \sigma \tag{21}$$

is a pseudo-Riemannian metric on B of signature $(2,2)$. There is also associated with η^3 a decomposition $\mathbb{S}D^* = \mathbb{S}D^+ \cup \mathbb{S}D^-$ and $\mathbb{R}^*$-isomorphisms

$$h : B \to \mathbb{S}D^+, \quad J \circ h : B \to D^-,$$

where h is given by equation (17) and J by equation (16), such that the conformal class of the induced metric on $\mathbb{S}D^*$ is independent of the choice of η^3; in fact, multiplying η^3 by a smooth non-vanishing function rescales the induced metric on $\mathbb{S}D^*$ by the same factor.

Proof. The proof is very similar to the CR case. Here are the formulas that differ:

$$\begin{aligned}
&\lambda_{10} = \lambda_1, \quad \lambda_{20} = -\lambda_2, \quad \lambda_{12} - \lambda_{21} = \lambda_3, \\
&\tilde{\alpha} = \alpha + \frac{3\lambda_1}{\lambda}\omega^1 - \frac{3\lambda_2}{\lambda}\omega^2 - \Big(\frac{\lambda_{12}+\lambda_{21}}{\lambda} + \frac{6\lambda_1\lambda_2}{\lambda^2}\Big)\omega^3, \\
&\dot{\alpha} = 3\dot{\lambda}_1\omega^1 - 3\dot{\lambda}_2\omega^2 - (\dot{\lambda}_{12} + \dot{\lambda}_{21})\omega^3, \\
&\mathrm{d}\dot{\alpha} \equiv -4(\dot{\lambda}_{12} + \dot{\lambda}_{21})\omega^1 \wedge \omega^2 \equiv (\dot{K} + 2\dot{\lambda}K)\omega^1 \wedge \omega^2 \pmod{\alpha, \omega^3}, \\
&\dot{K} = -2\dot{\lambda}K - 4(\dot{\lambda}_{12} + \dot{\lambda}_{21}), \\
&\dot{\mathrm{g}} = 2\dot{\lambda}\mathrm{g} + (\dot{\sigma} + 2\dot{\lambda}_1\omega^1 - 2\dot{\lambda}_2\omega^2). \qquad \blacksquare
\end{aligned}$$

Definition 3.2: A **chain** of a path geometry (M, L_1, L_2) is the projection to M of an unparametrized non-vertical null geodesic of the associated Fefferman conformal metric on $\mathbb{S}D^*$.

PROPOSITION 3.3:

(1) *The $\mathbb{R}^*$-action on $\mathbb{S}D^*$ is by conformal isometries.*
(2) *For every point in M, in every given direction transverse to D, there is a unique chain passing through this point in the given direction.*
(3) *The fibers of $\mathbb{S}D^* \to M$ are null geodesics and project to constant curves on M.*

Proof. (1) For every contact form η^3, the map $h : B \to \mathbb{S}D^+$ (by definition, a conformal isometry) is $\mathbb{R}^*$-equivariant, hence it is enough to verify that the pseudo-Riemannian metric on B given by equations (20)–(21) is $\mathbb{R}^*$-invariant. This follows from the $\mathbb{R}^*$-invariance of α, ω^3, K and the $\mathbb{R}^*$-equivariance

$$R_s^*\omega^1 = \omega^1/s, \quad R_s^*\omega^2 = s\omega^2.$$

(2) Let $x \in M$ and $v \in T_xM$, $v \notin D_x$. Pick a contact form η^3 and work on the associated bundle B. The fiber B_x consists of coframes $u = (u^1, u^2, u^3)$ on T_xM adapted to η^3, as in equations (14)–(15). We show that for every $u \in B_x$ there is a unique lift $\tilde{v} \in T_uB$ of v which is null. Now $\tilde{v}$ is a lift of v if and only if $\omega^i(\tilde{v}) = u^i(v)$, $i = 1, 2, 3$. It remains to determine $\sigma(\tilde{v})$. Now

$$\omega^3(\tilde{v}) = u^3(v) \neq 0$$

and, by formula (21), $\tilde{v}$ is null if and only if $\omega^1(\tilde{v})\omega^2(\tilde{v}) + \omega^3(\tilde{v})\sigma(\tilde{v}) = 0$, i.e., $\sigma(\tilde{v}) = -u^1(v)u^2(v)/u^3(v)$. This shows that $v \in T_xM$ has a unique null lift at $u \in B_x$. The null geodesic through u in the direction of $\tilde{v}$ projects to a chain through x in the direction of v. This proves existence of the required chain. As for uniqueness, we need to show that if we repeat the above at another point of B_x, say $s \cdot u \in B_x$, we obtain the same chain. We use the fact that s acts on B by isometries R_s, mapping $\tilde{v}$ to the unique null-lift of v at $s \cdot u$, and the null geodesic through u tangent to $\tilde{v}$ to the null geodesic through $s \cdot u$ in the direction of $(R_s)_* \tilde{v}$. Since R_s commutes with the projection $B \to M$, the two null geodesics project to the same chain in M.

(3) The vertical distribution of $B \to M$ is given by

$$\omega^1 = \omega^2 = \omega^3 = 0,$$

thus $\mathrm{g} = \omega^1 \cdot \omega^2 + \omega^3 \cdot \sigma$ restricted to the fibers vanishes, so these fibers are null curves. We proceed to show that they are null geodesics.

As shown in part (1) above, the principal $\mathbb{R}^*$-action on B is isometric. Let ζ denote an infinitesimal generator of this action (i.e., a nonzero vertical null Killing vector field on B). The fibers of $B \to M$ are the integral curves of ζ, hence to show that these fibers are null geodesics it is enough to show that $\nabla_\zeta \zeta = 0$, or in index notation,

$$\zeta^b \nabla_b \zeta^a = 0.$$

Lowering an index of $\nabla_b \zeta^a$ (using g), splitting $\nabla_b \zeta_a$ into its symmetric and antisymmetric parts, and contracting with ζ^b gives

$$\zeta^b \nabla_b \zeta_a = \zeta^b \cdot \frac{1}{2}(\nabla_a \zeta_b + \nabla_b \zeta_a) + \zeta^b \cdot \frac{1}{2}(\nabla_a \zeta_b - \nabla_b \zeta_a). \tag{22}$$

The quantity $\frac{1}{2}(\nabla_a \zeta_b + \nabla_b \zeta_a)$ in the first term is $(\mathcal{L}_\zeta g)_{ab}$, but, per part (1), g is ζ-invariant—that is, $\mathcal{L}_\xi g = 0$—and so the first term vanishes.

The quantity $\frac{1}{2}(\nabla_a \zeta_b - \nabla_b \zeta_a)$ in the second term is $(d\zeta^\flat)_{ab}$, so the second term is $-\iota_\zeta(d\zeta^\flat)$, where ι_ζ denotes interior multiplication by ζ. Since ζ generates the $\mathbb{R}^*$-action on $B \to M$ and α is a connection form thereon, $\alpha(\zeta)$ is a nonzero constant, and by rescaling ζ by a nonzero constant we may as well assume $\alpha(\zeta) = 3$. Lowering an index with g (equations (20)–(21)) then gives $\zeta^\flat = -\omega^3$, so the third equation of (18) yields $-\iota_\zeta(\mathrm{d}\zeta^\flat) = \iota_\zeta \mathrm{d}\omega^3 = \iota_\zeta(\omega^1 \wedge \omega^2) = 0$. ∎

Remark 3.4: In fact, chains come equipped with a preferred projective structure (see, e.g., [6, Theorem 5.3.7], which applies to all so-called parabolic contact structures), but we do not need that structure here.

3.1. Chains of $y'' = f(x, y, y')$. Here $\Sigma = J^0(\mathbb{R}, \mathbb{R}) = \mathbb{R}^2$, with coordinates (x, y), and $M = J^1(\mathbb{R}, \mathbb{R}) = \mathbb{R}^3$, with coordinates (x, y, p) and contact distribution $D = \mathrm{Ker}(\mathrm{d}y - p\,\mathrm{d}x)$. The paths in Σ are graphs of solutions $y(x)$ to $y'' = f(x, y, y')$, and their lifts to M are graphs of their first jets,

$$(x, y(x)) \mapsto (x, y(x), y'(x)).$$

So here

$$L_1 = \mathrm{Span}(\partial_p), \quad L_2 = \mathrm{Span}[\partial_x + p\partial_y + f(x, y, p)\partial_p].$$

We fix the contact form $\eta^3 := \mathrm{d}y - p\,\mathrm{d}x$. An adapted coframe on M, satisfying equations (14)–(15), is

$$\eta^1 := \mathrm{d}p - f\mathrm{d}x, \quad \eta^2 := -\mathrm{d}x, \quad \eta^3 := \mathrm{d}y - p\,\mathrm{d}x. \tag{23}$$

Any other adapted coframe is of the form

$$s \cdot \eta = (\eta^1/s, s\eta^2, \eta^3)^t, \quad s : M \to \mathbb{R}^*.$$

This defines an identification $\mathbb{R}^3 \times \mathbb{R}^* \to B$,

$$(x, y, p, s) \mapsto s \cdot \eta(x, y, p).$$

Under this identification,

$$\omega^1 = \eta^1/s, \quad \omega^2 = s\eta^2, \quad \omega^3 = \eta^3. \tag{24}$$

The following proposition was proved in [23, equation (31)] by solving the full equivalence problem for path geometry.

Proposition 3.5: *The Fefferman metric on $B = J^1(\mathbb{R}, \mathbb{R}) \times \mathbb{R}^*$ is*

$$\mathrm{g} = -\mathrm{d}x \cdot (\mathrm{d}p - f\mathrm{d}x) + \frac{1}{6}(\mathrm{d}y - p\,\mathrm{d}x) \cdot [4f_p\mathrm{d}x + f_{pp}(\mathrm{d}y - p\,\mathrm{d}x) - 4\mathrm{d}\tau], \tag{25}$$

where $\mathrm{d}\tau = \mathrm{d}s/s$.

Proof. Solving equations (18)–(19), with ω^i given by equations (23)–(24), we obtain

$$\alpha = -f_p\mathrm{d}x + \mathrm{d}\tau, \quad K = f_{pp} \implies \sigma = \frac{2}{3}f_p\mathrm{d}x + \frac{1}{6}f_{pp}(\mathrm{d}y - p\mathrm{d}x) - \frac{2}{3}\mathrm{d}\tau.$$

Using this in equations (20)–(21) gives the claimed formula. ∎

PROPOSITION 3.6: *The chains of the path geometry corresponding to a 2nd order ODE $y'' = f(x, y, y')$ are the curves in $J^1(\mathbb{R}, \mathbb{R})$ which are the graphs of solutions $(y(x), p(x))$ of the system*

$$y'' = f + f_p\Delta + \frac{1}{2}f_{pp}\Delta^2 + \frac{1}{6}f_{ppp}\Delta^3 \tag{26}$$

$$\begin{aligned} p'' = &- \frac{2(p'-f)^2}{\Delta} + f_p(3p' - 2f) + f_x + pf_y + [f_{pp}(p'-f) + 2f_y]\Delta \\ &+ \frac{1}{6}[f_{ppp}(p'-2f) - f_{xpp} + 4f_{yp} - pf_{ypp})]\Delta^2, \end{aligned} \tag{27}$$

where $\Delta = y' - p$.

Proof. Using the metric (25), we write the geodesic equations on B,

$$\begin{aligned} \ddot{x} &= \frac{1}{6}[(p\dot{x} - \dot{y})(f_{ppp}(\dot{y} - p\dot{x}) + 2\dot{x}f_{pp}) - 2\dot{x}^2 f_p - 4\dot{\tau}\dot{x}], \\ \ddot{y} &= \frac{1}{6}[2\dot{x}(pf_{pp}(p\dot{x} - \dot{y}) - p\dot{x}f_p - 2p\dot{\tau} + 3\dot{p}) - pf_{ppp}(\dot{y} - p\dot{x})^2], \\ \ddot{p} &= \frac{1}{6}[-p^3\dot{x}^2 f_{ypp} + 2p^2\dot{x}\dot{y}f_{ypp} + 4p^2\dot{x}^2 f_{yp} \\ &\quad - 2f((\dot{y} - p\dot{x})(f_{ppp}(\dot{y} - p\dot{x}) + 2\dot{x}f_{pp}) + 2\dot{x}^2 f_p + 4\dot{\tau}\dot{x}) \\ &\quad + 2\dot{p}f_{pp}(\dot{y} - p\dot{x}) - f_{xpp}(\dot{y} - p\dot{x})^2 - 8p\dot{x}\dot{y}f_{yp} - 6p\dot{x}^2 f_y \\ &\quad + 8\dot{p}\dot{x}f_p - p\dot{y}^2 f_{ypp} + 12\dot{x}\dot{y}f_y + 6\dot{x}^2 f_x + 4\dot{y}^2 f_{yp} + 4\dot{p}\dot{\tau}]. \end{aligned}$$

(We do not need the τ equation.) Next use formula (25) and the nullity condition to solve for $\dot{\tau}$,

$$\dot{\tau} = \frac{1}{4}(f_{pp}(\dot{y} - p\dot{x}) + 4\dot{x}f_p) - \frac{3}{2}\frac{\dot{x}(\dot{p} - f\dot{x})}{(\dot{y} - p\dot{x})},$$

then use this to eliminate $\dot{\tau}$ from the expression for $\ddot{p}$ (τ itself does not appear explicitly, because of the $\mathbb{R}^*$-invariance of the metric). Then substitute into $y'' = (\ddot{y}\dot{x} - \ddot{x}\dot{y})/\dot{x}^3, p'' = (\ddot{p}\dot{x} - \ddot{x}\dot{p})/\dot{x}^3$ the expressions for $\ddot{x}, \ddot{y}, \ddot{p}$ from the geodesic equations, and finally make the substitutions $\dot{y} = \dot{x}y', \dot{p} = \dot{x}p'$ to obtain the desired equations (all instances of $\dot{x}$ cancel out because the geodesic equation is homogeneously quadratic in velocities). ∎

3.2. CHAINS OF PROJECTIVE PATH GEOMETRIES. Here we prove Theorem 1, which was announced in the introduction. Recall that, by definition, a path geometry is projective if the paths are the (unparametrized) geodesics of a torsion-free affine connection.

Theorem 1: *A path geometry on a 2-dimensional manifold Σ is projective if and only if all chains on $\mathbb{P}T\Sigma$ project to the paths in Σ.*

Proof. This is a local statement so we can assume without loss of generality the situation studied in the previous subsection, i.e., the paths are given in the xy plane by graphs of solutions $y(x)$ of $y'' = f(x, y, y')$ for some smooth f, and the associated chains in xyp-space are the graphs of solutions $(y(x), p(x))$ to the chain equations (26)–(27) of Proposition 3.6. As is well known, such a path geometry is projective if and only if $f(x, y, p)$ is a polynomial in p of degree at most 3 (see [9], also Section 4 of [14]). The statement we are to prove therefore reduces to the following lemma:

Lemma 3.7: *Every solution $(y(x), p(x))$ of equations (26)–(27) satisfies*

$$y'' = f(x, y, y')$$

if and only if $f(x, y, p)$ is polynomial in p of degree at most 3.

We proceed with the proof of the lemma. Assume $f(x, y, p)$ is polynomial in p of degree ≤ 3. Then $f(x, y, y')$ is given by the cubic Taylor polynomial of f with respect to p:

$$f(x, y, y') = f + f_p(y' - p) + \frac{1}{2} f_{pp}(y' - p)^2 + \frac{1}{6} f_{ppp}(y' - p)^3, \tag{28}$$

where f and its derivatives on the right hand side are evaluated at (x, y, p). Now the right hand side of the last equation, evaluated at $y = y(x), y' = y'(x), p = p(x)$, is the right-hand side of the chain equation (26). It follows that if $(y(x), p(x))$ satisfy equations (26)–(27) then $y(x)$ satisfies $y''(x) = f(x, y(x), y'(x))$, as needed.

Conversely, suppose $f(x, y, p)$ is not polynomial in p of degree ≤ 3. Then there is a neighborhood $U \subset \mathbb{R}^3$ such that for all $(x, y, p), (x, y, y') \in U$, with $y' \neq p$, equation (28) does not hold. It follows that the chains in this neighborhood do not project to solutions of $y'' = f(x, y, y')$. ∎

Remark 3.8: One should also be able to prove Theorem 1 using the general machinery of parabolic geometry concerning correspondence spaces [6, Section 4.4] and canonical curves [6, Section 5.3] in a way that may be readily generalizable to other types of parabolic geometries and families of curves. Such a proof would take us too far afield here, so we will take up this approach elsewhere.

4. Examples of path geometries and their chains

In this section we illustrate the general theory of the previous section by determining explicitly the chains of some homogeneous path geometries. First, the flat path geometry on $\mathbb{R}P^2$, admitting an 8-dimensional symmetry group, then 3 of the items of Tresse's classification [25] of 'submaximal' path geometries, i.e., those admitting a 3-dimensional group of symmetries. In each case we exploit the symmetry to reduce the chain equations to determining null geodesics on a group with respect to a left-invariant pseudo-Riemannian metric. Then a well-known procedure reduces the equations to the Euler equations on the dual of the Lie algebra of the group and are integrable.

4.1. The flat path geometry. Here $M \subset \mathbb{R}P^2 \times (\mathbb{R}P^2)^*$ is the set of incident point-line pairs (q, ℓ) (equivalently, the manifold $F_{1,2}$ of full flags in $\mathbb{R}^3$) and $L_1, L_2 \subset TM$ are the tangents to the fibers of the projections onto the first and second factor (respectively).

Proposition 4.1: *For each non-incident pair $(q_*, \ell_*) \in \mathbb{R}P^2 \times (\mathbb{R}P^2)^* \setminus M$ consider the set of incident pairs $(q, \ell) \in M$ such that $q \in \ell_*, q_* \in \ell$. This is a chain in M and all chains in M are of this form. See Figure 2.*

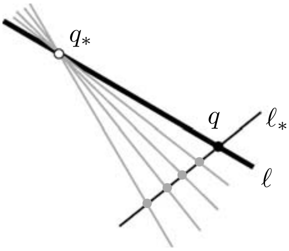

Figure 2. Chains of the flat path geometry (straight lines).

To prove it, note that $\mathrm{GL}_3(\mathbb{R})$ acts naturally on (M, L_1, L_2). We look for a 3-dimensional subgroup $G \subset \mathrm{GL}_3(\mathbb{R})$ acting on M with an open orbit. Fixing a point $m_0 = (q_0, \ell_0) \in M$ yields two left-invariant line fields $L_1, L_2 \subset TG$, given by their value $(L_1)_{\mathrm{id}}, (L_2)_{\mathrm{id}} \subset \mathfrak{g}$, the Lie algebras of the stabilizers of q_0, ℓ_0 (resp.). It is then easy to find left-invariant adapted coframes on G describing L_1, L_2 and the associated Fefferman metric. We consider two such G: the Heisenberg group and the Euclidean group.

4.1.1. *First proof of Proposition 4.1: via the Heisenberg group.* Let H be the set of matrices of the form

$$\begin{pmatrix} 1 & z & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}. \tag{29}$$

Its Lie algebra $\mathfrak{h}$ consists of matrices of the form

$$\begin{pmatrix} 0 & x^1 & x^3 \\ 0 & 0 & x^2 \\ 0 & 0 & 0 \end{pmatrix}, \quad x^i \in \mathbb{R}. \tag{30}$$

Let θ^i be the left-invariant 1-form on H whose value at id $\in$ H is x^i, $i = 1, 2, 3$. Then

$$\Theta := \begin{pmatrix} 0 & \theta^1 & \theta^3 \\ 0 & 0 & \theta^2 \\ 0 & 0 & 0 \end{pmatrix}$$

is the left-invariant Maurer–Cartan form on H, satisfying $\mathrm{d}\Theta = -\Theta \wedge \Theta$, from which we get

$$\mathrm{d}\theta^1 = \mathrm{d}\theta^2 = 0, \quad \mathrm{d}\theta^3 = -\theta^1 \wedge \theta^2. \tag{31}$$

Identify $\mathbb{R}^2$ with an affine plane in $\mathbb{R}^3$, $(x, y) \mapsto (y, x, 1)$. It is H-invariant, and the resulting affine action on $\mathbb{R}^2$ is $(x_0, y_0) \mapsto (x_0 + x, y_0 + y + zx_0)$. This action is transitive on $\mathbb{R}^2$ and transitive and free on the set M of incident pairs (q, ℓ), where $q \in \mathbb{R}^2$ and $\ell \subset \mathbb{R}^2$ is a non-vertical line through q. There are H-invariant line fields $L_1, L_2 \subset TM$, where L_1 (resp. L_2) is tangent to the fibers of the projection $(q, \ell) \mapsto q$ (resp. $(q, \ell) \mapsto \ell$).

Let $q_0 = (0, 0), \ell_0 = \{y = 0\}$ (the real axis). Let (X_1, X_2, X_3) be the (left-invariant) frame on H dual to $(\theta^1, \theta^2, \theta^3)$. Then the Lie algebras $(L_1)_{\mathrm{id}}, (L_2)_{\mathrm{id}}$ of the stabilizers of q_0, ℓ_0 are spanned by X_1, X_2 (resp.). Thus,

$$D = L_1 \oplus L_2 = (\theta^3)^\perp, \quad L_1 = \{\theta^2, \theta^3\}^\perp, \quad L_2 = \{\theta^1, \theta^3\}^\perp,$$

with an adapted coframe

$$\eta^1 := \theta^1, \quad \eta^2 := \theta^2, \quad \eta^3 := -\theta^3.$$

Solving the structure equations (18)–(19), we get $\alpha = \theta^4,\ K = 0$, where

$$\theta^4 = (\mathrm{d}s)/s$$

(the Maurer–Cartan form on $\mathbb{R}^*$), which gives, using equations (20)–(21),

$$\sigma = -(2/3)\theta^4$$

and

$$\mathrm{g} = \theta^1 \cdot \theta^2 + \frac{2}{3}\theta^3 \cdot \theta^4. \tag{32}$$

Lemma 4.2: *Null geodesics of* (32), *projected to* H *and passing through* $\mathrm{id} \in \mathrm{H}$ *at* $t = 0$, *are of the form*

$$x = b(1 - e^{-ct}), \quad y = -ab(1 - e^{-ct}), \quad z = a(e^{ct} - 1), \quad a, b, c \in \mathbb{R}.$$

They correspond to chains $(q_t, \ell_t) \in M$, *passing through* (q_0, ℓ_0) *at* $t = 0$, *where* q_t *moves along the line* ℓ_* *through* q_0 *of slope* $-a$, *and* ℓ_t *is a line through* q_t *and* $q_* = (b, 0)$.

Proof. The metric (32) is a left-invariant metric on $G = \mathrm{H} \times \mathbb{R}^*$, with an inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$

$$A = \frac{1}{6}\begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

The geodesic flow on T^*G projects via left translation to the Euler equations on $\mathfrak{g}^*$, $\dot{P} = \mathrm{ad}^*_{A^{-1}P}P$, where $\mathrm{ad}^*_X = (\mathrm{ad}_X)^t \in \mathrm{End}(\mathfrak{g}^*)$, $X \in \mathfrak{g}$, $P \in \mathfrak{g}^*$ and $\mathrm{ad}_X Y = [X, Y]$. These are the Hamiltonian equations $\dot{P} = \{H, P\}$ with respect to the standard Lie–Poisson structure on $\mathfrak{g}^*$, where $H = \frac{1}{2}(P, A^{-1}P)$. See [2, page 66]. Equivalently, $\dot{X} = A^{-1}\mathrm{ad}^*_X AX$. To write these down explicitly with respect to our bases, we first represent $X \in \mathfrak{g}$ and $\mathrm{ad}^*_X \in \mathrm{End}(\mathfrak{g}^*)$ by 4×4 matrices

$$X = \begin{pmatrix} 0 & x^1 & x^3 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^4 \end{pmatrix}, \quad \mathrm{ad}^*_X = (\mathrm{ad}_X)^t = \begin{pmatrix} 0 & 0 & -x^2 & 0 \\ 0 & 0 & x^1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\dot{X} = A^{-1}\mathrm{ad}^*_X AX$ becomes

$$\dot{x}^1 = \frac{2}{3}x^1x^4, \quad \dot{x}^2 = -\frac{2}{3}x^2x^4, \quad \dot{x}^3 = \dot{x}^4 = 0. \tag{33}$$

The general solution, with $H = (AX, X)/2 = x^1x^2/2 + x^3x^4/3 = 0$ (we are interested in the zero level set because we are computing the null geodesics), is

$$x^1 = ae^{ct}, \quad x^2 = be^{-ct}, \quad x^3 = -\frac{ab}{c}, \quad x^4 = \frac{3c}{2}, \tag{34}$$

where $a, b, c \in \mathbb{R}$, $c \neq 0$. (In addition to these solutions there are some fixed points, which we now ignore.)

Now let $g(t) \in H \times \mathbb{R}^*$ be a null geodesic, with

$$g(t) = \begin{pmatrix} 1 & z & y & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{pmatrix}, \quad x, y, z, s \in \mathbb{R},\ s \neq 0.$$

Then $X = g^{-1}\dot{g} \in \mathfrak{g}$ is given by (34). Explicitly,

$$\dot{x} = x^2 = b\,e^{-ct}, \quad \dot{y} - z\dot{x} = x^3 = -\frac{ab}{c}, \quad \dot{z} = x^1 = a\,e^{ct} \tag{35}$$

(we do not need the s equation). Change the time variable to $\tau = ct$, denoting derivative with respect to τ by $(\ )'$ and renaming the constants, $a \mapsto a/c, b \mapsto b/c$, we get

$$x' = be^{-\tau}, \quad y' - zx' = -ab, \quad z' = ae^{\tau}. \tag{36}$$

Consider chains through id $\in$ H, i.e., $x_0 = y_0 = z_0 = 0$. Then $z = a(e^\tau - 1)$, hence $y' = zx' - ab = -ab\,e^{-\tau}$. The solution of (36) is then

$$x = b(1 - e^{-\tau}), \quad y = -ab(1 - e^{-\tau}), \quad z = a(e^\tau - 1).$$

Thus (x, y) traces a line of slope $-a$ through the origin, and each line of slope z through (x, y) passes through $(b, 0)$. ∎

4.1.2. *Second proof of Proposition 4.1: via the Euclidean group.* Here

$$M = \mathbb{P}(T\mathbb{R}^2) = \mathbb{R}^2 \times \mathbb{P}(\mathbb{R}^2)$$

is the set of pairs (q, ℓ) with ℓ a line through q, $L_1 \subset TM$ is tangent to the fibers of the projection onto the first factor and similarly for L_2. The group SE_2 of orientation-preserving isometries of $\mathbb{R}^2$ acts transitively on M, with stabilizer $\mathbb{Z}_2$ (reflection about a point), preserving L_1, L_2. Fixing a point $(q_0, \ell_0) \in M$ identifies M with $\mathrm{SE}_2/\mathbb{Z}_2$, and hence equips SE_2 with left-invariant line fields L_1, L_2 given by a pair of 1-dimensional subspaces $(L_1)_{\mathrm{id}}, (L_2)_{\mathrm{id}}$, the Lie algebras of the stabilizers of q_0, ℓ_0 (resp.).

Identify $\mathbb{R}^2 = \mathbb{C}$ with the affine plane $\mathbf{z}_2 = 1$ in $\mathbb{C}^2$, $\mathbf{z} \mapsto (\mathbf{z}, 1)$; then SE_2 is identified with the subgroup of $\mathrm{GL}_2(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} e^{i\theta} & \mathbf{z} \\ 0 & 1 \end{pmatrix}, \quad \mathbf{z} \in \mathbb{C},\ \theta \in \mathbb{R}. \tag{37}$$

Its Lie algebra $\mathfrak{se}_2$ consists of matrices of the form

$$\begin{pmatrix} ix^1 & x^2 + ix^3 \\ 0 & 0 \end{pmatrix}, \quad x^i \in \mathbb{R}. \tag{38}$$

Let θ^j be the left-invariant 1-form on SE_2 whose value at id is x^j, $j = 1, 2, 3$. Then

$$\Theta := \begin{pmatrix} i\theta^1 & \theta^2 + i\theta^3 \\ 0 & 0 \end{pmatrix}$$

is the left-invariant Maurer–Cartan form on SE_2, satisfying $\mathrm{d}\Theta = -\Theta \wedge \Theta$, from which we get

$$\mathrm{d}\theta^1 = 0, \quad \mathrm{d}\theta^2 = \theta^1 \wedge \theta^3, \quad \mathrm{d}\theta^3 = -\theta^1 \wedge \theta^2. \tag{39}$$

Let X_1, X_2, X_3 be the left-invariant vector fields on SE_2 dual to $\theta^1, \theta^2, \theta^3$. Let $q_0 = 0, \ell_0 = \mathbb{R}$ (the real axis). Then the Lie algebras of the stabilizers of q_0, ℓ_0 are spanned by X_1, X_2 (resp.). Thus

$$D = L_1 \oplus L_2 = (\theta^3)^\perp, \quad L_1 = \{\theta^2, \theta^3\}^\perp, \quad L_2 = \{\theta^1, \theta^3\}^\perp,$$

with an adapted coframe

$$\eta^1 := \theta^1, \quad \eta^2 := \theta^2, \quad \eta^3 := -\theta^3.$$

Solving the structure equations (18)–(19), we get $\alpha = \theta^4$, $K = 0$, where $\theta^4 = (\mathrm{d}s)/s$ (the Maurer–Cartan form on $\mathbb{R}^*$), which gives, using equations (20)–(21), $\sigma = -(2/3)\theta^4$ and

$$\mathrm{g} = \theta^1 \cdot \theta^2 + \frac{2}{3}\theta^3 \cdot \theta^4. \tag{40}$$

This is a left-invariant metric on $G = \mathrm{SE}_2 \times \mathbb{R}^*$, with an inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$

$$A = \frac{1}{6} \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

The geodesic flow on T^*G projects via left translation to the Euler equations on $\mathfrak{g}^*$, $\dot P = \mathrm{ad}^*_{A^{-1}P}P$, where $\mathrm{ad}^*_X = -(\mathrm{ad}_X)^t \in \mathrm{End}(\mathfrak{g}^*)$. These are the Hamiltonian equations $\dot P = \{H, P\}$ with respect to the standard Lie–Poisson structure on $\mathfrak{g}^*$, where $H = \frac{1}{2}(P, A^{-1}P)$. See [2, p. 66]. To write these down explicitly with respect to our bases, we first represent $X \in \mathfrak{g}$ and $\mathrm{ad}^*_X \in \mathrm{End}(\mathfrak{g}^*)$ by 4×4 matrices

$$X = \begin{pmatrix} 0 & -x^1 & x^2 & 0 \\ x^1 & 0 & x^3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^4 \end{pmatrix}, \quad \mathrm{ad}^*_X = (\mathrm{ad}_X)^t = \begin{pmatrix} 0 & x^3 & -x^2 & 0 \\ 0 & 0 & x^1 & 0 \\ 0 & -x^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\dot P = \mathrm{ad}^*_{A^{-1}P}P$ becomes

$$\dot P_1 = -2P_1P_3 + 3P_2P_4, \quad \dot P_2 = 2P_2P_3, \quad \dot P_3 = -2P_2^2, \quad \dot P_4 = 0, \tag{41}$$

with constants of motion (in addition to P_4),

$$H = \frac{1}{2}(P, A^{-1}P) = 2P_1P_2 + 3P_3P_4 = 0, \quad k = (P_2)^2 + (P_3)^2.$$

Let us use polar coordinates in the P_2P_3-plane:

$$P_2 = r\cos\phi, \quad P_3 = r\sin\phi.$$

Then

$$\dot\phi = -2r\cos\phi, \quad P_1 = c\tan\phi, \quad P_4 = -2c/3, \quad c = const., \quad r = const.$$

Now let $g(t) \in \mathrm{SE}_2 \times \mathbb{R}^*$ be a null geodesic, with

$$g(t) = \begin{pmatrix} e^{i\theta} & \mathbf{z} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix}, \quad \mathbf{z} \in \mathbb{C},\ \theta, s \in \mathbb{R}^*.$$

Let $X = g^{-1}\dot g \in \mathfrak{g}$. Then $P = AX$ satisfies Euler equations (41). Explicitly,

$$\dot\theta = x^1 = 2P_2 = -\dot\phi, \quad \dot{\mathbf{z}} = e^{i\theta}(x^2 + ix^3) = e^{i\theta}(2P_1 + i3P_4) = 2ce^{i\theta}(\tan\phi - i).$$

(The s equation is omitted; it will not be used.) Assume, without loss of generality, that $g(0) = \mathrm{id}$, i.e., $\theta(0) = 0$ and $\mathbf{z}(0) = 0$, so $\theta = \phi_0 - \phi$. We reparametrize $g(t)$ by ϕ, denote derivative with respect to ϕ by $(\ )'$, and get

$$\mathbf{z}' = i(c/r)e^{i\phi_0}\dot{\mathbf{z}}/\dot\phi = i(c/r)e^{i\phi_0}\sec^2\phi.$$

Integrating yields $\mathbf{z} = i(c/r)e^{i\phi_0}\tan\phi$. Now we rotate the chain by $-\phi_0$, reflect about the x-axis and rename c, so that

$$\mathbf{z} = ic\tan\phi, \quad \theta = \phi, \quad c \in \mathbb{R}. \tag{42}$$

This corresponds to a chain (q_ϕ, ℓ_ϕ), where q_ϕ moves along $\ell^* =$ the y axis and ℓ_ϕ is the line connecting $q_* = (-c, 0)$ with q_ϕ.

4.2. Circles of fixed radius. Here $M \subset \mathbb{C} \times \mathbb{C}$ is the set of pairs of points (p, q) with $|p - q| = 1$, $L_1, L_2 \subset TM$ are tangent to the fibers of the projection onto the first (resp. second) factor. The first projection maps the fibers of the second projection to the set of plane circles of radius 1. The group SE_2 of orientation-preserving isometries of $\mathbb{C} = \mathbb{R}^2$ acts transitively and freely on M, preserving L_1, L_2. We use the same notation for this group as in Section 4.1.2. Let $p_0 = 0, q_0 = 1$. Then the Lie algebras $(L_1)_{\mathrm{id}}, (L_2)_{\mathrm{id}}$ of the stabilizers of these points are spanned by $X_1, X_1 - X_3$ (resp.). Thus

$$D = L_1 \oplus L_2 = (\theta^2)^\perp, \quad L_1 = \{\theta^2, \theta^3\}^\perp, \quad L_2 = \{\theta^2, \theta^1 + \theta^3\}^\perp,$$

with an adapted coframe

$$\eta^1 := \theta^3, \quad \eta^2 := \theta^1 + \theta^3, \quad \eta^3 := -\theta^2.$$

Solving the structure equations (18)–(19), we get $\alpha = \theta^2 + \theta^4$, $K = -1$, where $\theta^4 = (\mathrm{d}s)/s$ (the Maurer–Cartan form on $\mathbb{R}^*$), which gives, using equations (20)–(21), $\sigma = -\theta^2/2 - 2\theta^4/3$ and

$$\mathrm{g} = \frac{1}{2}(\theta^2)^2 + (\theta^3)^2 + \theta^1 \cdot \theta^3 + \frac{2}{3}\theta^2 \cdot \theta^4. \tag{43}$$

This is a left-invariant metric on $G = \mathrm{SE}_2 \times \mathbb{R}^*$, with an inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$

$$A = \frac{1}{6}\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 6 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

The geodesic flow on T^*G projects via left translation to the Euler equations on $\mathfrak{g}^*$, $\dot{P} = \mathrm{ad}^*_{A^{-1}P}P$. To write these down explicitly with respect to our bases, we first represent $X \in \mathfrak{g}$ and $\mathrm{ad}^*_X \in \mathrm{End}(\mathfrak{g}^*)$ by 4×4 matrices

$$X = \begin{pmatrix} 0 & -x^1 & x^2 & 0 \\ x^1 & 0 & x^3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^4 \end{pmatrix}, \quad \mathrm{ad}^*_X = (\mathrm{ad}_X)^t = \begin{pmatrix} 0 & x^3 & -x^2 & 0 \\ 0 & 0 & x^1 & 0 \\ 0 & -x^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\dot P = \mathrm{ad}^*_{A^{-1}P} P$ become

$$\begin{aligned} &\dot P_1 = 2P_1P_2 - 3P_3P_4, \quad \dot P_2 = 2P_3(P_3 - 2P_1), \\ &\dot P_3 = -2P_2(P_3 - 2P_1), \quad \dot P_4 = 0, \end{aligned} \tag{44}$$

with constants of motion (in addition to P_4),

$$\begin{aligned} H &= \frac{1}{2}(P, A^{-1}P) = -2P_1^2 + 2P_1P_3 + 3P_2P_4 - \frac{9}{4}M_4^2, \\ r^2 &= (P_2)^2 + (P_3)^2. \end{aligned}$$

We make the following change of variables:

$$y = 4P_1 - 2P_3, \quad P_2 = r\cos\phi, \quad P_3 = -r\sin\phi, \quad P_4 = c/3. \tag{45}$$

Then (44) reduces to

$$\dot\phi = -y, \quad \dot y = 4r(c - r\cos\phi)\sin\phi, \quad c, r \in \mathbb{R},\ r \geq 0, \tag{46}$$

and the nullity condition $H = 0$ becomes

$$y^2 = 8cr\cos\phi + 4r^2\sin^2\phi - 2c^2. \tag{47}$$

Remark 4.3: Equations (46) can be written as a single Newton type second order ODE, $\ddot\phi = f(\phi)$, where $f(\phi) = 4r(r\cos\phi - c)\sin\phi$. As usual, one can write $f(\phi) = -U'(\phi)$, with

$$U(\phi) = -4cr\cos\phi - 2r^2\sin^2\phi.$$

Then $\dot\phi^2/2 + U(\phi)$ is constant along solutions of $\ddot\phi = f(\phi)$ ('conservation of energy'). Equation (47) fixes the value of this constant.

Now let $g(t) \in \mathrm{SE}_2 \times \mathbb{R}^*$ be a null geodesic, with

$$g(t) = \begin{pmatrix} e^{i\theta} & \mathbf{z} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix}, \quad \mathbf{z} \in \mathbb{C},\ \theta, s \in \mathbb{R}^*.$$

Let $X = g^{-1}\dot{g} \in \mathfrak{g}$. Then $P = AX$ satisfies (44). Explicitly,

$$\dot{\mathbf{z}} = e^{i\theta}(x^2 + ix^3) = e^{i\theta}(3P_4 + i2P_1), \quad \dot{\theta} = x^1 = -4P_1 + 2P_3.$$

Using the change of variables (45), we get

$$\dot{\mathbf{z}} = e^{i\theta}[c - i(\dot{\phi}/2 + r\sin\phi)], \quad \dot{\theta} = \dot{\phi},$$

where $\phi(t)$ satisfies equations (46)–(47). For a fixed $\phi(t)$ these equations are invariant under rigid motions (adding a constant angle to θ, rotating $\mathbf{z}$ by this angle and translating $\mathbf{z}$ by some constant vector). So we can assume, without loss of generality, that $\theta = \phi$. Hence

$$\dot{\mathbf{z}} = e^{i\theta}[c - i(\dot{\theta}/2 + r\sin\theta)], \quad \dot{\theta}^2 = 8cr\cos\theta + 4r^2\sin^2\theta - 2c^2.$$

Next we use the scaling invariance, $t \mapsto \lambda t,\ c \mapsto \lambda c,\ r \mapsto \lambda r$, to assume $r = 1$. We can also use the reflection symmetry $t \mapsto -t, \theta \mapsto \theta + \pi, \mathbf{z} \mapsto -\mathbf{z}, c \mapsto -c$ to assume that $c \geq 0$. Thus every chain, up to a rigid motion and reparametrization, is a solution to

$$\dot{\mathbf{z}} = e^{i\theta}[c - i(\dot{\theta}/2 + \sin\theta)], \tag{48}$$

$$(\dot{\theta})^2 = 8c\cos\theta + 4\sin^2\theta - 2c^2, \tag{49}$$

with $c \in \mathbb{R},\ c \geq 0$.

Lemma 4.4: *Let $F(\theta, c) = 8c\cos\theta + 4\sin^2\theta - 2c^2$ (the right-hand side of equation (49)). Then $F \geq 0$ has a solution if and only if $|c| \leq 4$. For every $c \in [0, 4]$ the set of $\theta \in [-\pi, \pi]$ such that $F(\theta, c) \geq 0$ is an interval $[-\theta_{\max}, \theta_{\max}]$, where $\theta_{\max} = \cos^{-1}(c - \sqrt{c^2/2 + 1}) \in [0, \pi]$. For $c \in (0, 4)$ every solution $\theta(t)$ of (49) oscillates between $-\theta_{\max}$ and $\theta_{\max}$. If $c = 0$ then $\lim\theta$ is 0 or π as $t \to \pm\infty$. If $c = 4$ then $\theta \equiv 0$.*

Proof. We write $F = -4x^2 + 8cx + 4 - 2c^2$, where $x = \cos\theta$. The roots of this polynomial are $x_\pm = c \pm \sqrt{1 + c^2/2}$ and $F > 0$ in the interval (x_-, x_+). To be able to solve for θ we need $[x_-, x_+]$ to intersect the interval $[-1, 1]$. It is elementary to show that this occurs if and only if $|c| \leq 4$. ∎

Proposition 4.5: *Every chain of the path geometry of circles of radius 1 in the Euclidean plane, up to an affine reparametrization and rigid motion, is given by a unique solution of equations (48)–(49) with $c \in [0, 4)$, $\mathbf{z}(0) = \theta(0) = 0$ (for $c = 0$ one should take $\theta(0) \neq 0, \pi$).*

See Figure 3. The projection of the chains on the Euclidean plane (the curves $\mathbf{z}(t)$) look like inflectional elastica, but they are not (checked numerically).

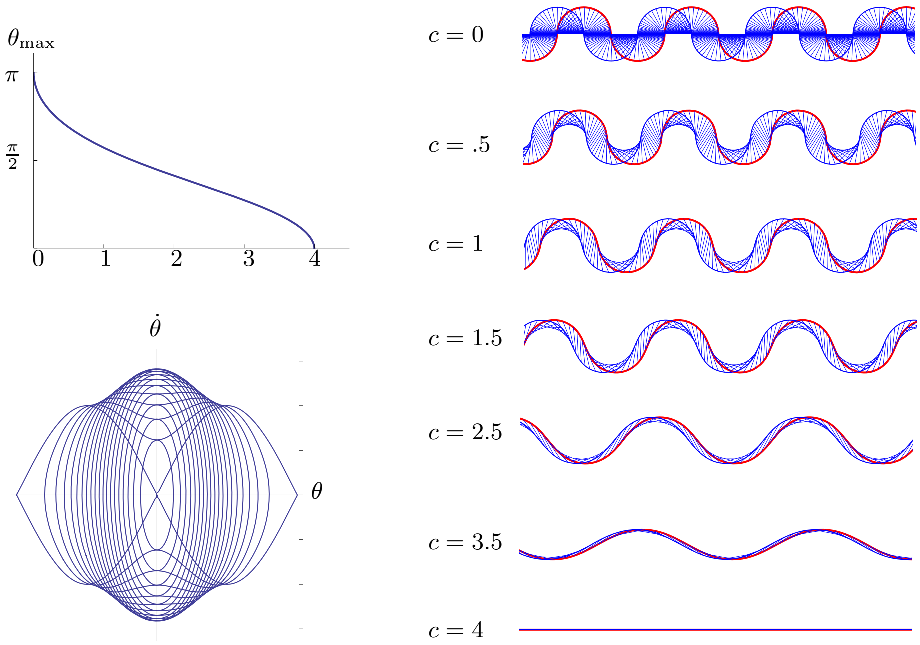

Figure 3. Chains of circles path geometry (solutions of equations (48)–(49)). Top left: plot of the maximum amplitude of oscillation of θ as a function of the chain parameter $c \in [0, 4]$. Bottom left: phase curves of equation (49) for various c values. Right: each red curve is the projection of the chain on the Euclidean plane. The blue curve represents the projection of the chain on the dual plane; it is formed by joining the tips of the unit vectors in the direction θ at each point $\mathbf{z}$ of the red curve (the thin blue lines).

Further properties/questions about these chains:

(1) From the pictures, $\mathbf{z}(t) + e^{i\theta(t)}$ (the red curve) is obtained from $\mathbf{z}(t)$ by translation and parameter shift. Presumably, this comes out of equations (48)–(49). Is this a manifestation of the self-duality of this path geometry? How exactly?

(2) One should be able to write explicit solutions of equations (48)–(49) using elliptic functions. See [24].

Note. One can write down an explicit general solution for the case $c=0$ without any special functions, and one can verify analytically that the arcs are semicircles. So, as embedded submanifolds they are C^1 but not C^2 at inflection points. In particular, the chain ODE is not satisfied at these inflection points.

(3) Equation (49) is the equation of a pendulum under a strange force law:

$$f(\theta) = 4(\cos\theta - c)\sin\theta,$$

with special initial conditions: $\theta(0) = 0, \dot\theta(0) = 8c - c^2$ (for $c = 0$ it is the homoclinic solution of the pendulum equation $\ddot\theta = 2\sin 2\theta$). Is there a good mechanical/geometrical interpretation of this motion?

(4) The chains of this geometry project to a 1-parameter family of curves in $\mathbb{R}^2$ (up to rigid motion). Is there a simple geometric description of this family? Our first guess was elastica but it is not the case.

(5) In the pictures, there are points along $\mathbf{z}(t)$ at which $\theta(t)$ is the direction of the tangent $\dot{\mathbf{z}}(t)$ (the inflection points of the red curves on the right of Figure 3). Is this phenomenon unavoidable?

4.3. Hooke ellipses of fixed area. The manifold

$$M = \{(x, y, E, F, G) \in \mathbb{R}^5 \mid Ex^2 + 2Fxy + Gz^2 = 1,\ EG - F^2 = 1,\ E > 0\}$$

parametrizes the set of incident pairs $(\mathbf{r}, \mathcal{E})$, where $\mathbf{r} = (x, y)^t \in \mathbb{R}^2 \setminus 0$ and $\mathcal{E}$ is an ellipse centered at the origin (a 'Hooke ellipse') of area π.

Proposition 4.6: *The path geometry in $\mathbb{R}^2 \setminus 0$ of Hooke ellipses of fixed area is projective (the paths are the unparametrized geodesics of a torsion-free affine connection).*

Proof. As mentioned before, this is equivalent to showing that the associated ODE $y'' = f(x, y, y')$ is cubic in y'. Let

$$\mathbb{H} = \{(E, F, G) \mid EG - F^2 = 1, E > 0\}$$

be the path space. We parametrize $\mathbb{H}$ by the upper half-plane $\mathbb{R}^2_+ = \{(a, b) \mid b > 0\}$,

$$(a, b) \mapsto \frac{1}{b}(1, -a, a^2 + b^2). \tag{50}$$

Hooke ellipses of area π are then given by equations of the form

$$x^2 - 2axy + (a^2 + b^2)y^2 = b, \quad a, b \in \mathbb{R}, b > 0. \tag{51}$$

Assuming $y = y(x)$ in this equation and taking two derivatives with respect to x, we get

$$x - a(y + xy') + (a^2 + b^2)yy' = 0,$$
$$1 - a(2y' + xy'') + (a^2 + b^2)[(y')^2 + yy''] = 0.$$

Eliminating a, b from the last 3 equations and solving for y'', we obtain

$$y'' = (xy' - y)^3.$$

Another proof, more direct, consists of showing that Hooke ellipses of area π are the (unparametrized) geodesics of a Riemannian metric in $\mathbb{R}^2 \setminus 0$, given in polar coordinates by

$$ds^2 = dr^2/\Delta^2 + r^2 d\theta^2/\Delta, \quad \Delta = 1 + cr^2 + r^4, \quad c \in \mathbb{R}.$$

See [3] for yet another proof, via equivalence with the path geometry of Kepler ellipses of fixed major axis, which is projective since these are geodesics of the Jacobi–Maupertuis metric of the Kepler problem. ∎

Fefferman metric. Let $L_1 \subset TM$ be the tangents to the fibers of the projection on the first component, $(q, \mathcal{E}) \mapsto q$, and similarly for L_2. The group $\mathrm{SL}_2(\mathbb{R})$ acts transitively and freely on M via its standard linear action on $\mathbb{R}^2$, preserving L_1, L_2. Fixing a point $(q_0, \mathcal{E}_0) \in M$ identifies M with $\mathrm{SL}_2(\mathbb{R})$, and L_1, L_2 with two left-invariant line fields on $\mathrm{SL}_2(\mathbb{R})$, given at $\mathrm{id} \in \mathrm{SL}_2(\mathbb{R})$ by the Lie algebras of the stabilizers of $q_0, \mathcal{E}_0$, respectively.

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ consists of matrices of the form

$$\begin{pmatrix} x^1 & x^2 \\ x^3 & -x^1 \end{pmatrix}, \quad x^i \in \mathbb{R}.$$

The left-invariant $\mathfrak{sl}_2(\mathbb{R})$-valued Maurer–Cartan form on $\mathrm{SL}_2(\mathbb{R})$ is

$$\Theta = g^{-1}\mathrm{d}g = \begin{pmatrix} \theta^1 & \theta^2 \\ \theta^3 & -\theta^1 \end{pmatrix}. \tag{52}$$

The Maurer–Cartan equation $\mathrm{d}\Theta = -\Theta \wedge \Theta$ gives

$$\mathrm{d}\theta^1 = -\theta^2 \wedge \theta^3, \quad \mathrm{d}\theta^2 = -2\theta^1 \wedge \theta^2, \quad \mathrm{d}\theta^3 = 2\theta^1 \wedge \theta^3. \tag{53}$$

Fix $q_0 := (1,0)^t$, $\mathcal{E}_0 := \{x^2+y^2=1\}$. Then

$$(L_1)_{\mathrm{id}} = \mathrm{Span}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad (L_2)_{\mathrm{id}} = \mathrm{Span}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad D = L_1 \oplus L_2 = \mathrm{Ker}(\theta^1).$$

An adapted coframe is thus

$$\eta^1 = \theta^2 + \theta^3, \quad \eta^2 = \theta^3, \quad \eta^3 := -\theta^1.$$

We use this coframe to trivialize the associated $\mathbb{R}^*$-structure $B \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^*$ and put the standard coordinate s on the $\mathbb{R}^*$ factor. The associated 1-forms on B are

$$\omega^1 = \frac{1}{s}(\theta^2+\theta^3), \quad \omega^2 = s\theta^3, \quad \omega^3 = -\theta^1.$$

Solving the structure equations (18)–(19), we get

$$\alpha = 2\theta^1 + \theta^4, \quad a_1 = 4/s^2, \quad a_2 = 0, \quad K = -2,$$

where $\theta^4 = (\mathrm{d}s)/s$ (the Maurer–Cartan form on $\mathbb{R}^*$), which gives, using equations (20)–(21), $\sigma = -\theta^1 - (2/3)\theta^4$ and

$$\mathrm{g} = (\theta^1)^2 + (\theta^3)^2 + \theta^2\theta^3 + \frac{2}{3}\theta^1\theta^4. \tag{54}$$

Hooke chains (null geodesics of the Fefferman metric). The pseudo-Riemannian metric (54) is a left-invariant metric on the Lie group

$$G := \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^*.$$

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \times \mathbb{R}$ be its Lie algebra and $A : \mathfrak{g} \to \mathfrak{g}^*$ the inertia operator corresponding to the quadratic form (54); that is, $\mathrm{g}(X,Y) = (AX)Y$, $X, Y \in \mathfrak{g}$. Then

$$A = \frac{1}{6}\begin{pmatrix}6 & 0 & 0 & 2\\ 0 & 0 & 3 & 0\\ 0 & 3 & 6 & 0\\ 2 & 0 & 0 & 0\end{pmatrix}$$

(with respect to the basis $\{\theta^i\}$ and its dual). As in previous examples, the geodesic flow on T^*G projects to $\dot{P} = \mathrm{ad}^*_{A^{-1}P}P$ on $\mathfrak{g}^*$, the Hamiltonian equations with respect to the standard Lie–Poisson structure on $\mathfrak{g}^*$ with Hamiltonian $H = \frac{1}{2}(P, A^{-1}P)$. To write these down explicitly, we first represent $X \in \mathfrak{g}$

and $\mathrm{ad}^*_X \in \mathrm{End}(\mathfrak{g}^*)$ by the matrices

$$X = \begin{pmatrix} x^1 & x^2 & 0 \\ x^3 & -x^1 & 0 \\ 0 & 0 & x^4 \end{pmatrix}, \quad \mathrm{ad}^*_X = \begin{pmatrix} 0 & -2x^2 & 2x^3 & 0 \\ -x^3 & 2x^1 & 0 & 0 \\ x^2 & 0 & -2x^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\dot{P} = \mathrm{ad}^*_{A^{-1}P} P$ becomes

$$\begin{aligned} \dot{P}_1 &= 8(P_2)^2, & \dot{P}_2 &= 2P_2(3P_4 - P_1), \\ \dot{P}_3 &= 2P_1(P_3 - 2P_2) - 6P_3P_4, & \dot{P}_4 &= 0, \end{aligned} \tag{55}$$

with constants of motion P_4, k, H, where

$$k = (P_1)^2 + 4P_2P_3, \quad H = 2P_2P_3 - 2(P_2)^2 + 3P_1P_4 - 9P_4^2/2 = 0. \tag{56}$$

Note that k is a Casimir of $\mathfrak{g}^*$ coming from the Killing form of $\mathfrak{sl}_2(\mathbb{R})$. We set $H = 0$ since we are looking for null geodesics. Next we make the following change of variables:

$$sP_1 = b(c + \sin\phi), \quad P_2 = \frac{b}{2}\cos\phi, \quad P_3 = b\Big(\cos\phi - \frac{p}{2}\Big), \quad P_4 = \frac{bc}{3}.$$

We have $k - 2H = b^2$, hence b is constant. Since $P_4 = bc/2$ is constant c is constant as well. Equations (55)–(56) then reduce to

$$\dot{\phi} = 2b\cos\phi, \quad p = \cos\phi + c(c + 2\sin\phi)\sec\phi. \tag{57}$$

Next let $g(t) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^*$ be a null geodesic, with

$$g(t) = \begin{pmatrix} x & z & 0 \\ y & w & 0 \\ 0 & 0 & s \end{pmatrix}, \quad x, y, z, w, s \in \mathbb{R},\ s \neq 0,\ xw - yz = 1.$$

Let $X = g^{-1}\dot{g} \in \mathfrak{g}$. Then $P = AX$ satisfies equations (55). Explicitly,

$$\begin{aligned} \dot{x} &= x^1x + x^3z = b[cx + (\cos\phi)z], & \dot{z} &= x^2x - x^1z = -b[p\,x + cz], \\ \dot{y} &= x^1y + x^3w = b[cy + (\cos\phi)w], & \dot{w} &= x^2y - x^1w = -b[p\,y + cw], \end{aligned}$$

where p, ϕ are given by equation (57). Denote $\mathbf{r} := (x, y)$, $\mathbf{h} := (z, w) \in \mathbb{R}^2$, then the last system is

$$\dot{\mathbf{r}} = b[c\mathbf{r} + (\cos\phi)\mathbf{h}], \quad \dot{\mathbf{h}} = -b[p\mathbf{r} + c\mathbf{h}]. \tag{58}$$

Lemma 4.7: *ϕ is twice the centro-affine arclength of the projection of the chain to the Hooke plane (the $\mathbf{r}$ plane).*

Proof. $\mathbf{r}, \mathbf{h}$ are the columns of a matrix in $\mathrm{SL}_2(\mathbb{R})$, hence $[\mathbf{r}, \mathbf{h}] = 1$. It then follows from equations (58) that $[\mathbf{r}, \mathrm{d}\mathbf{r}/\mathrm{d}\phi] = [\mathbf{r}, \dot{\mathbf{r}}/\dot{\phi}] = [\mathbf{r}, \mathbf{h}/2] = 1/2$. ∎

Let us reparametrize the chains by $\tau := \phi/2$ (the centro-affine arclength) and denote derivative with respect to τ by $(\ )'$. Equations (58) now become

$$\begin{aligned} \mathbf{r}' &= c(\sec 2\tau)\mathbf{r} + \mathbf{h}, \\ \mathbf{h}' &= -[1 + c(c + 2(\sin 2\tau))\sec^2 2\tau]\mathbf{r} - c(\sec 2\tau)\mathbf{h}. \end{aligned} \tag{59}$$

Lemma 4.8:

$$\mathbf{r}'' = -\mathbf{r}.$$

Proof. Straightforward calculation from equations (59). ∎

Thus, combined with $[\mathbf{r}, \mathbf{r}'] = 1$ (Lemma 4.7), each Hooke chain projects to a Hooke ellipse of area π in the $\mathbf{r}$ plane, as expected from Proposition 4.6 and Theorem 1.

Proposition 4.9: *Every chain in $\mathrm{SL}_2(\mathbb{R})$ of the path geometry of Hooke ellipses of area π, up to left translation, is of the form*

$$\mathbf{r} = e^{i\tau}, \quad \mathbf{h} = e^{i\tau}(-c\sec(2\tau) + i)$$

(using complex notation), for some $c \in \mathbb{R}$, $c \neq 0$. See Figure 4.

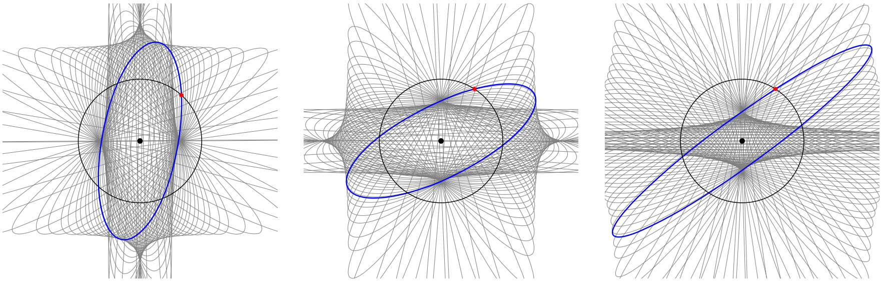

Figure 4. Hooke's chains, given by Proposition 4.9, for $c = -1, 1, 2$.

Proof. $\mathrm{SL}_2(\mathbb{R})$ acts transitively on Hooke ellipses of area π, hence the projection of the chain to the $\mathbf{r}$ plane can be brought to the unit circle. Parametrized by centro affine arc length, it is $\mathbf{r} = e^{i\tau}$. Then the 1st equation of (59) implies the formula for $\mathbf{h}(\tau)$. For $c = 0$ this formula produces a curve tangent to the contact distribution, which is excluded. ∎

4.4. Horocycles in the hyperbolic plane. The space of Hooke ellipses is $\mathbb{H} = \{(E, F, G) \mid EG - F^2 = 1, E > 0\}$, the hyperboloid model of the hyperbolic plane. The curves in $\mathbb{H}$ of constant (hyperbolic) curvature 1 are called **horocycles** and are the sections of $\mathbb{H}$ by planes parallel to a generator of the cone $EG - F^2 = 0$. In the upper half-plane model these are (Euclidean) circles tangent to the real axis.

Lemma 4.10: *For each fixed $(x, y) \in \mathbb{R}^2 \setminus 0$, the set of Hooke ellipses passing through (x, y) is a horocycle in $\mathbb{H}$. This defines a bijection between the punctured plane $\mathbb{R}^2 \setminus 0$ and the space of horocycles in $\mathbb{H}$.*

Proof. For each $(x, y) \in \mathbb{R}^2 \setminus 0$, equation (51),

$$x^2 - 2axy + (a^2 + b^2)y^2 = b,$$

defines in the upper half-plane $\{(a, b) \mid b > 0\}$ either the circle of radius $\frac{1}{2y^2}$ centered at $(\frac{x}{y}, \frac{1}{2y^2})$ if $y \neq 0$, or the horizonal line $b = x^2$ if $y = 0$. These are precisely all the horocycles of the upper half plane model of the hyperbolic plane. ∎

It follows that the horocycle path geometry in $\mathbb{H}$ is dual to the path geometry in $\mathbb{R}^2 \setminus 0$ of Hooke ellipses of fixed area. Thus we can use the analysis of the previous section to determine the projection of the chains to $\mathbb{H}$.

Proposition 4.11: *Each chain of the horocycle path geometry, up to the action of $\mathrm{SL}_2(\mathbb{R})$, projects to a curve in the hyperbolic plane, given in the upper half-plane model $\{(x, y) \mid y > 0\}$ by*

$$\begin{aligned}(x^2 + y^2)^2 - [4cx + (c^2 + 4)y](x^2 + y^2) + (6c^2 - 2)x^2 + 2c^3xy + 6y^2 \\ -4c(c^2 - 1)x - (c^4 - 3c^2 + 4)y + (c^2 - 1)^2 = 0\end{aligned} \tag{60}$$

where $c \neq 0$. See Figure 5. This curve is the projection of a chain in $\mathrm{SL}_2(\mathbb{R})$, the solution to equations (59) that passes through $\mathrm{id} \in \mathrm{SL}_2(\mathbb{R})$. The projection of this chain to the Hooke plane is the Hooke ellipse $(x - cy)^2 + y^2 = 1$. The horocycles along this chain, in the upper half-plane model, all pass through $(c, 1)$, the point corresponding to this Hooke ellipse. The chains corresponding to c and $-c$ are congruent via an outer automorphism of $\mathrm{SL}_2(\mathbb{R})$ (conjugation by $\mathrm{diag}(-1, 1) \in \mathrm{GL}_2(\mathbb{R})$), acting by reflection about the y-axis.

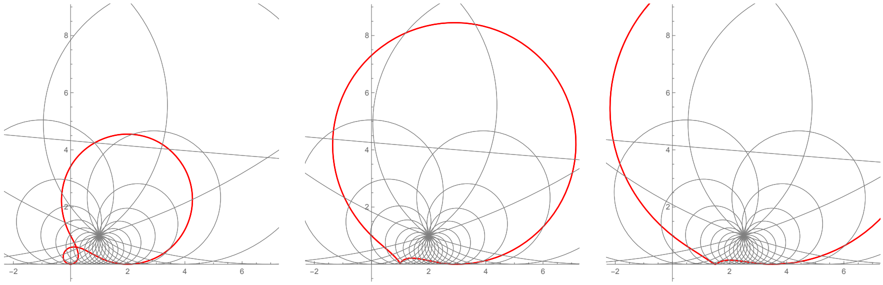

Figure 5. Horocycle chains, given by Proposition 4.11, projected to the hyperbolic plane (the upper half plane model), are rational bicircular quartics. Left: Crunodal (one node), $|c| < 2$. Middle: Cuspidal (one cusp), $c = 2$. Right: Acnodal (smooth), $|c| > 2$.

Proof. Using $\mathbf{r}'' = -\mathbf{r}$ and $\mathbf{r}' = c(\sec 2\tau)\mathbf{r} + \mathbf{h}$ (Lemma 4.8 and equation (59)), the chain $g(\tau)$ in $\mathrm{SL}_2(\mathbb{R})$ with $g(0) = \mathrm{id}$ is

$$g(\tau) = \begin{pmatrix} \cos\tau + c\sin\tau & -\sin\tau(\sec(2\tau)c^2 + \tan(2\tau)c + 1) \\ \sin\tau & \cos\tau - c\sec(2\tau)\sin\tau \end{pmatrix}. \tag{61}$$

The projection of this chain to $\mathbb{H}$ is obtained by acting by $g(\tau)$ on the point in $\mathbb{H}$ corresponding to the Hooke ellipse $\mathcal{E}_0 = \{x^2 + y^2 = 1\}$. One can check that the parametrization of $\mathbb{H}$ by the upper half-plane in equation (50) is $\mathrm{SL}_2(\mathbb{R})$-equivariant, so one can act instead by $g(\tau)$ via fractional linear transformations on $(0, 1)$, the point in the upper half-plane corresponding to $\mathcal{E}_0$. Reverting to $\phi = 2\tau$, the outcome is

$$(x, y) = \frac{(c^2[(c + 2\sin\phi)\cos\phi - c - \sin\phi],\ -2\cos^2\phi)}{-2\cos^2\phi + c(c + 2\sin\phi)\cos\phi - c^2}.$$

Eliminating ϕ in the above equation (we used Maple for this), one obtains equation (60). ∎

Remark 4.12: The curves of Proposition 4.11 are examples of bicircular quartics, a remarkable class of plane curves introduced by J. Casey in 1871 [8]. See [27] for a modern exposition. They have many equivalent geometric and algebraic definitions, the simplest being the inversion of a conic (with respect to a circle).

References

[1] V. I. Arnol'd, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Grundlehren der mathematischen Wissenschaften, Vol. 250, Springer, New York, 1988.

[2] V. I. Arnol'd and A. B. Givental, *Symplectic geometry*, in *Dynamical Systems. IV* Encyclopaedia of Mathematical Sciences, Vol. 4, Springer, Berlin, 2001, pp. 1–138.

[3] G. Bor and H. Jacobowitz, *Left-invariant CR structures on 3-dimensional Lie groups*, Complex Analysis and its Synergies **7** (2021), Article no. 23.

[4] G. Bor and C. Jackman, *Revisiting Kepler: new symmetries of an old problem*, Arnold Mathematical Journal **9** (2023), 267–299.

[5] D. Burns Jr., K. Diederich and S. Shnider, *Distinguished curves in pseudoconvex boundaries*, Duke Mathematical Journal **44** (1977), 407–431.

[6] A. Čap and J. Slovák, *Parabolic Geometries. I*, Mathematical Surveys and Monographs, Vol. 154, American Mathematical Society, Providence, RI, 2009.

[7] A. Čap and V. Žádník, *On the geometry of chains*, Journal of Differential Geometry **82** (2009), 1–33.

[8] J. Casey, *On bicircular quartics*, Transactions of the Royal Irish Academy **24** (1871), 457–569.

[9] E. Cartan, *Sur les variétés à connexion projective*, Bulletin de la Société Mathématique de France **52** (1924), 205–241.

[10] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes. Part I*, Annali di Matematica Pura ed Applicata **11** (1932), 17–90; **Part II**, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze **1** (1932), 333–354.

[11] A. Castro and R. Montgomery, *The chains of left-invariant Cauchy–Riemann structures on $SU(2)$*, Pacific Journal of Mathematics **238** (2008), 41–71.

[12] J. H. Cheng, T. Marugame, V. S. Matveev and R. Montgomery, *Chains in CR geometry as geodesics of a Kropina metric*, Advances in Mathematics **350** (2019), 973–999.

[13] S. S. Chern and J. Moser, *Real hypersurfaces in complex manifolds*, Acta Mathematica **133** (1974), 219–271.

[14] B. Doubrov and B. Komrakov, *The geometry of second-order ordinary differential equations*, `https://arxiv.org/abs/1602.00913`.

[15] J. Douglas, *The general geometry of paths*, Annals of Mathematics **29** (1928), 143–168.

[16] F. A. Farris, *An intrinsic construction of Fefferman's CR metric*, Pacific Journal of Mathematics **123** (1986), 33–45.

[17] C. L. Fefferman, *Monge–Ampére equations, the Bergman kernel, and geometry of pseudoconvex domains*, Annals of Mathematics **103** (1976), 395–416.

[18] K. Hughen, *The geometry of subriemannian 3-manifolds*, PhD. Thesis, Duke University, Durham, NC, 1995, `https://pdfs.semanticscholar.org/4069/84ef45565eae1bfe4241e70eb3ed8b60f88b.pdf`.

[19] T. A. Ivey and J. M. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, Graduate Studies in Mathematics, Vol. 61, American Mathematical Society, Providence, RI, 2003.

[20] H. Jacobowitz, *An Introduction to CR Structures*, Mathematical Surveys and Monographs, Vol. 32, American Mathematical Society, Providence, RI, 1990.

[21] B. Kruglikov, *Point classification of second order ODEs: Tresse classification revisited and beyond*, in *Differential Equations: Geometry, Symmetries and Integrability*, Abel Symposia, Vol. 5, Springer, Berlin–Heidelberg, 2009, pp. 199-221.

[22] J. M. Lee, *The Fefferman metric and pseudo-Hermitian invariants*, Transactions of the American Mathematical Society **296** (1986), 411–429.

[23] P. Nurowski and G. Sparling, *Three-dimensional Cauchy–Riemann structures and second-order ordinary differential equations*, Classical and Quantum Gravity **20** (2003), 4995–5016.

[24] G. Pastras, *Four Lectures on Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics*, `https://arxiv.org/abs/1706.07371`

[25] A. M. L. Tresse, *Détermination des invariants ponctuels de l'équation difféentielle ordinaire du second ordre* $y''\omega(x, y, y')$, Hirzel, Leipzig, 1896.

[26] S. Webster, *Pseudo-Hermitian structures on a real hypersurface*, Journal of Differential Geometry **13** (1978), 25–41.

[27] T. R. Werner, *Rational families of circles and bicircular quartics*, PhD. thesis, Friedrich-Alexander-Universitaet, Erlangen-Nuernberg, 2012,
`https://opus4.kobv.de/opus4-fau/files/2301/ThomasWernerDissertation.pdf`